## REPORT DOCUMENTATION PAGE

| | |
|---|---|
| 1a. REPORT SECURITY CLASSIFICATION<br>Unclassified | 1b. RESTRICTIVE MARKINGS<br>None |
| 2a. SECURITY CLASSIFICATION AUTHORITY | 3. DISTRIBUTION/AVAILABILITY OF REPORT<br>Approved for public release; distribution unlimited |
| 2b. DECLASSIFICATION/DOWNGRADING SCHEDULE | |
| 4. PERFORMING ORGANIZATION REPORT NUMBER(S)<br>NORDA Report 170 | 5. MONITORING ORGANIZATION REPORT NUMBER(S)<br>NORDA Report 170 |
| 6. NAME OF PERFORMING ORGANIZATION<br>Naval Ocean Research and Development Activity | 7a. NAME OF MONITORING ORGANIZATION<br>Naval Ocean Research and Development Activity |
| 6c. ADDRESS *(City, State, and ZIP Code)*<br>Ocean Science Directorate<br>NSTL, Mississippi 39529-5004 | 7b. ADDRESS *(City, State, and ZIP Code)*<br>Ocean Science Directorate<br>NSTL, Mississippi 39529-5004 |
| 8a. NAME OF FUNDING/SPONSORING ORGANIZATION<br>Office of Naval Research<br>8b. OFFICE SYMBOL *(If applicable)* | 9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER<br>Code 132 |
| 8c. ADDRESS *(City, State, and ZIP Code)*<br>Washington, D.C. | 10. SOURCE OF FUNDING NOS<br>PROGRAM ELEMENT NO. 62435N; PROJECT NO. RM35G86; TASK NO. M0G; WORK UNIT NO. 13237E |
| 11. TITLE *(Include Security Classification)*<br>Energy Transfers for Multilayer Hydrodynamic Ocean Models: General Formulation and Specific Examples | |
| 12. PERSONAL AUTHOR(S)<br>Donna Wells Blake | |
| 13a. TYPE OF REPORT<br>Final<br>13b. TIME COVERED<br>From _____ To _____ | 14. DATE OF REPORT *(Yr., Mo., Day)*<br>August 1987<br>15 PAGE COUNT<br>102 |
| 16. SUPPLEMENTARY NOTATION | |
| 17 COSATI CODES<br>FIELD / GROUP / SUB GR | 18 SUBJECT TERMS *(Continue on reverse if necessary and identify by block number)*<br>energetics, ocean dynamics, primitive equations, layered numerical models |

19 ABSTRACT *(Continue on reverse if necessary and identify by block number)*

Energy analysis is a common technique for examining the physical behavior of fluid systems. The specifications of energies and their time variations are not unique, but depend on the dynamical model being studied. A dynamical model consists of a set of equations giving the mathematical representation of the relevant physical laws, together with the particular assumptions and approximations being applied. Here two dynamical models, useful in studying ocean circulations, are treated: the inviscid multilayer hydrodynamic model and the inviscid multilayer reduced-gravity model. The two models are identical except that the velocity is required to be zero in the bottom layer of the reduced-gravity model. The reduced-gravity model retains most of the physical properties of the finite depth model but has considerably fewer computational requirements.

Two different formulations of the energetics are obtained: layer-layer and layer-interface. In both, the kinetic energy per unit area of a layer is specified. In the layer-layer formulation the available potential energy per unit area for a layer is specified while the layer-interface formulation uses the available potential energy per unit area associated with an interface. In addition, the types of energy transfer possible are different for the two formulations. For the layer-layer formulation, kinetic energy and available

| | |
|---|---|
| 20. DISTRIBUTION/AVAILABILITY OF ABSTRACT<br>UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ■ DTIC USERS ☐ | 21. ABSTRACT SECURITY CLASSIFICATION<br>Unclassified |
| 22a. NAME OF RESPONSIBLE INDIVIDUAL<br>Donna Wells Blake | 22b. TELEPHONE NUMBER *(Include Area Code)*<br>(601) 688-4727<br>22c OFFICE SYMBOL<br>Code 322 |

REPORT DOCUMENTATION PAGE

19. ABSTRACT (Continued)

potential energy are exchanged within a layer, and available potential energy is exchanged between adjacent layers. There is no transfer of kinetic energy between layers. For the layer-interface formulation energy is transferred between the kinetic energy in a layer and the available potential energy in an adjacent interface, above or below the layer. In each formulation the magnitudes of the energy transfer terms differ for the two models. In addition, for the layer-interface formulation the reduced-gravity model, but not the finite depth model, permits the exchange of available potential energy between consecutive interfaces. This latter exchange is a consequence of the required coupling of interface deviations in reduced-gravity models. Finally, both formulations are valid mathematically and physically, so the selection of one depends on what types of energy transfers the user wishes to examine.

The energetics, or energy analysis, is presented for a general multilayer case for both models and both formulations. Also, the energetics in both formulations are considered for specific cases: the one-, two-, and three-layer cases for the finite depth model and the one and two active layer cases for the reduced-gravity model.

Naval Ocean Research and Development Activity
August 1987
NORDA Report 170

# Energy Transfers for Multilayer Hydrodynamic Ocean Models: General Formulation and Specific Examples

Donna Wells Blake
Ocean Sensing and Prediction Division
Ocean Science Directorate

# Foreword

Knowledge of the space and time evolution of environmental parameters in the ocean can significantly improve acoustic surveillance capabilities, performance of weapons systems, search and rescue planning, and other aspects of naval operations. Observations alone cannot provide this information and must be supplemented by the use of numerical forecast models. The Naval Ocean Research and Development Activity is designing and developing a hierarchy of numerical forecast models, including multilayer hydrodynamic models, for Navy use in predicting the ocean environment. The energy analysis of forecast results is a major diagnostic technique in testing forecast results and must be formulated specifically for each numerical forecast model.

**A. C. Esau, Captain, USN**
**Commanding Officer, NORDA**

# Executive summary

Energy analysis is a common technique for examining the physical behavior of fluid systems. The specifications of energies and their time variations are not unique but depend on the dynamical model being studied. A dynamical model consists of a set of equations giving the mathematical representation of the relevant physical laws, together with the particular assumptions and approximations being applied. Here two dynamical models, useful in studying ocean circulations, are treated: the inviscid multilayer hydrodynamic model and the inviscid multilayer reduced-gravy hydrodynamic model or, for brevity, the finite depth model and the reduced-gravity model. The two models are identical except that the velocity is required to be zero in the bottom layer of the reduced-gravity model. The reduced-gravity model retains most of the physical properties of the finite depth model but has considerably fewer computational requirements.

Two different formulations of the energetics are obtained: layer-layer and layer-interface. In both, the kinetic energy per unit area of a layer is specified. In the layer-layer formulation the available potential energy per unit area for a layer is specified while the layer-interface formulation uses the available potential energy per unit area associated with an interface. In addition, the types of energy transfer possible are different for the two formulations. For the layer-layer formulation kinetic energy and available potential energy are exchanged within a layer and available potential energy is exchanged between adjacent layers. There is no transfer of kinetic energy between layers. For the layer-interface formulation energy is transferred between the kinetic energy in a layer and the available potential energy in an adjacent interface, above or below the layer. In each formulation the magnitudes of the energy transfer terms differ for the two models. In addition, for the layer-interface formulation the reduced-gravity model, but not the finite depth model, permits the exchange of available potential energy between consecutive interfaces. This latter exchange is a consequence of the required coupling of interface deviations in reduced-gravity models. Finally, both formulations are valid mathematically and physically so the selection of one depends on what types of energy transfers the user wishes to examine.

The energetics, or energy analysis, is presented for a general multilayer case for both models and both formulations. Also, the energetics in both formulations are considered for specific cases: the one-, two-, and three-layer cases for the finite depth model and the one and two active layer cases for the reduced-gravity model.

# Acknowledgments

This work was supported by the Office of Naval Technology under program element PE 62435N, project RM35G86, block manager, Dr. Gerald B. Morris. The author would like to acknowledge the valuable assistance provided by task area manager, Dr. George Heburn (NORDA), Dr. Harley Hurlburt (NORDA), and Dr. Alan Wallcraft of JAYCOR. Special thanks are due Ms. Leona Cole (NORDA) for her expert typing, Robert Woodyard (JAYCOR) for his graphics support, and the staff of Code 125P (NORDA) for their special efforts in publishing a complex manuscript.

# Contents

# Energy Transfers for Multilayer Hydrodynamic Ocean Models: General Formulation and Specific Examples

## Glossary

| | |
|---|---|
| $A_i$ | available potential energy per unit area in layer i |
| $\tilde{A}_i$ | available potential energy associated with the interface above layer i |
| d/dt | $\partial/\partial t + \partial/\partial x + \partial/\partial y$, total time derivative |
| $F_i$ | energy flux divergence for layer i |
| f | Coriolis parameter |
| g | acceleration due to gravity |
| $H_i$ | thickness of layer i when system is at rest |
| $h_i$ | actual layer thickens |
| $\eta_i$ | interface deviation at top of layer i |
| i,j | subscripts referring to a particular layer or interface |
| $\hat{i},\hat{j}$ | unit vectors in the x, y directions |
| $K_i$ | kinetic energy per unit area for layer i |
| $\hat{n}_i$ | unit vector normal to side walls in layer i |
| N | total number of layers or interfaces |
| $p_i$ | depth-dependent pressure for layer i |
| $\hat{p}_i$ | depth-independent pressure for layer i |
| $\tilde{p}_i$ | modified depth-independent pressure for layer i, used in reduced-gravity model |
| $P_i$ | potential energy per unit area for layer i |
| $\bar{P}_i$ | minimum potential energy per unit area for layer i |
| $\rho_i$ | density in layer i |
| RHS | right hand side |

| | |
|---|---|
| $S_i$ | area of side walls in layer i |
| $t$ | time |
| $\vec{v}$ | $= ui + vj$ |
| w.r.t. | with respect to |
| $x$ | east-west distance, positive to the east |
| $y$ | north-south distance, positive to the north |
| $z$ | vertical distance, positive upward and zero at the area-averaged ocean top |
| $\tilde{z}_i$ | actual depth at bottom of layer i |
| $\vec{\nabla}$ | $= i\, \partial/\partial x + j\, \partial/\partial y$ |
| $\partial/\partial t$ | local time derivative |
| $\langle\,\rangle$ | angle brackets indicate an average over the area of the system |

## PART A: INTRODUCTION

Numerical forecasting of the ocean environment requires first a consistent set of equations that express the relevant physical laws for the dynamics of geophysical fluids together with appropriate initial and boundary conditions and then a rapid and accurate numerical method for integrating the equations in space and time. The physical laws for the ocean, in general, include conservation laws for mass, momentum, and salinity, an equation of state and a thermodynamic relation. A unique mathematical representation of these physical laws does not exist. Furthermore, due to the complexity of these equations, various assumptions and approximations may be introduced which simplify the equations or even reduce the number of equations. The numerical integration method is then tailored to the chosen set of equations. This set of equations together with the particular integration method is referred to as a numerical forecast model. Any such model may be subject to different sets of boundary and initial conditions provided each set used is consistent with the given model.

The validity of the numerical forecast generated by a particular numerical model with a selected set of boundary and initial conditions is judged in part by comparison with observed conditions in the ocean region. Improving the forecast requires changing the boundary and initial conditions applied or changing the numerical model itself. Such changes must be based on a thorough understanding of the dominant physical mechanisms incorporated in the model and in the applied boundary and initial conditions. Analysis of the energetics in the forecast region is one technique used to fathom the physical processes present.

An introduction to energy analysis in geophysical fluid dynamics can be found in a number of textbooks (e.g., Dutton, 1976; Gill, 1982; Holton, 1972; Pedlosky,1987; and Phillips, 1977). The three types of energy considered are kinetic, potential, and internal, but the specific formulation of each is dependent on the given set of equations being used. Thus, the energetics analysis for a numerical model is dependent on the chosen set of equations and must be rederived for any change in that set. One of the objectives of the Naval Ocean Research and Development Activity's (NORDA) FY 86-87 Tactical Oceanography Project is to develop a general energetics analysis for the multilayer hydrodynamic numerical model developed originally by Hurlburt and Thompson (1980), hereafter referred to as H & T.

The development of a general energy analysis for a given model begins with the formulation of the types of energy present. Next, the change of these energies with respect to time (w.r.t.) is obtained. This step usually requires considerable mathematical manipulation to obtain a physically reasonable formulation for the transfer of energy between the various types of energy present and between the various regions in the fluid domain. Finally, the energy analysis for numerical models usually involves some type of space and/or time averaging. This report includes only the first two steps which are interrelated and lengthy; the averaging techniques will be presented in a later report.

The set of equations appropriate for an inviscid multilayer hydrodynamic model is derived in the Appendix. These equations, valid for all layers, form

the basis for the general energy analysis and are listed in Table 1 for ready reference. For some research investigations, an additional assumption is made: namely, that the bottom layer, N, is assumed to be of infinite depth. The numerical model with this additional assumption is called an inviscid multilayer reduced-gravity hydrodynamic model. For brevity, the original model is called the finite depth model and the second model is called the reduced-gravity model. The energy analysis for both models is developed, but a derivation of the changes in the basic set of equations for the reduced-gravity model is delayed until section B.5.

Table 1. Basic equations for the finite depth model

$$\frac{\partial u_i}{\partial t} + \vec{v}_i \cdot \vec{\nabla} u_i - f v_i = - \frac{1}{\rho_i} \left( \frac{\partial p_i}{\partial x} \right) \tag{1}$$

$$\frac{\partial v_i}{\partial t} + \vec{v}_i \cdot \vec{\nabla} v_i + f u_i = - \frac{1}{\rho_i} \left( \frac{\partial p_i}{\partial y} \right) \tag{2}$$

$$\frac{\partial h_i}{\partial t} + \vec{\nabla} \cdot \left( h_i \vec{v}_i \right) = 0 \tag{3}$$

$$p_i = \rho_i g \left( \eta_1 - z \right) - \sum_{j=1}^{i} \left( \rho_i - \rho_j \right) g h_j \tag{4}$$

The set of equations in Table 1 differ from those found in H & T in several respects. First, the equations in H & T are in transport form and those in Table 1 are in momentum form. The momentum form, rather than the transport form, is used in this report because velocity, not transport, is used in formulating kinetic energy. Second, the equations in Table 1 do not contain any external forcing, such as winds, nor do they have any viscous effects or dissipation included. Such sources and sinks determine the resulting circulation in hydrodynamic models but their inclusion in any energy analysis is straightforward: the sources increase the kinetic energy and the sinks decrease the kinetic energy. The difficult part of any energy analysis is determining a mathematically consistent and physically reasonable formulation for the tranfers of energy. Once this formulation is determined, adding the source and sink terms is quite simple. Therefore, the source and sink terms are omitted in this report to simplify the mathematical analysis included.

The energy analysis for layered models is developed in section B.1 without reference to the number of layers present. Various ways of treating

The energy analysis for layered models is developed in section B.1 without reference to the number of layers present. Various ways of treating the kinetic energy and the available potential energy are presented in the following sections in Part B and equations showing the change of these energies w.r.t. time are derived. One significant result of this investigation is that there are a variety of ways to express the change w.r.t. time of both the kinetic and available potential energies. Two formulations are selected for further consideration: the first or layer-layer uses the available potential energy in a layer while the second or layer-interface treats available potential energy associated with each interface. In both treatments the kinetic energy for a layer is used.

Many forecasts made with multilayer models actually use only one or two layers. Also the energy relations for the top (bottom) layer differ from those for the internal layers because there is no layer above (below) with which to exchange energy. Therefore, in Parts C (layer-layer formulation) and D (layer-interface formulation) the specific energy relations for one, two, and three layers are given for the finite depth model and for one active and two active layers for the reduced-gravity model. Each section in Parts C and D is designed to stand alone; therefore, there is some overlap in the descriptions presented in each section.

This report is designed for two purposes: first, to delineate the energy analysis formulations best suited to the inviscid multilayer hydrodynamic model and to the inviscid multilayer reduced-gravity hydrodynamic model; second, to serve as a training manual for those scientists who are unfamiliar with the techniques of energy analysis used with forecast models. The second purpose dictates that a considerable amount of explanation and mathematical detail be included that generally is eliminated for brevity.

Part B, Section 1

Kinetic, potential, and internal are the three types of energy commonly considered in the dynamics of geophysical fluids. In hydrodynamic models, the temperature is not a variable; therefore, internal energy is not considered. Furthermore, the models being analyzed are layer models so it seems reasonable to examine kinetic energy and potential energy in a layer, rather than for the entire volume of the ocean region. The structure of the layers in the finite depth model is shown in Figure 1.

The kinetic energy density in layer i is defined as $\rho_i \left(u_i^2 + v_i^2\right)/2$. The column integrated kinetic energy density or the kinetic energy per unit area in a layer i is $K_i$ where

$$K_i \equiv \frac{1}{2} h_i \rho_i \left(u_i^2 + v_i^2\right) . \tag{1.1}$$

All symbols are defined in the Glossary.

The change w.r.t. time of $K_i$ can be derived from the basic equations derived in the Appendix and listed in Table 1:

$$\left(\frac{\partial u_i}{\partial t}\right) + u_i \left(\frac{\partial u_i}{\partial x}\right) + v_i \left(\frac{\partial u_i}{\partial y}\right) - f v_i = - \frac{1}{\rho_i} \left(\frac{\partial p_i}{\partial x}\right) \tag{1.2}$$

$$\left(\frac{\partial v_i}{\partial t}\right) + u_i \left(\frac{\partial v_i}{\partial x}\right) + v_i \left(\frac{\partial v_i}{\partial y}\right) + f u_i = - \frac{1}{\rho_i} \left(\frac{\partial p_i}{\partial y}\right) \tag{1.3}$$

$$\left(\frac{\partial h_i}{\partial t}\right) + u_i \left(\frac{\partial h_i}{\partial x}\right) + v_i \left(\frac{\partial h_i}{\partial y}\right) + h_i \left(\frac{\partial u_i}{\partial x}\right) + h_i \left(\frac{\partial v_i}{\partial y}\right) = 0 \tag{1.4}$$

$$p_i = \rho_i g \left(\eta_1 - z\right) - \sum_{j=1}^{i} \left(\rho_i - \rho_j\right) g h_j \tag{1.5}$$

Multiply (1.2) by $h_i \rho_i u_i$ and (1.3) by $h_i \rho_i v_i$. Add the resulting equations to obtain

$$h_i \frac{\partial}{\partial t} \left[\frac{1}{2} \rho_i \left(u_i^2 + v_i^2\right)\right] + h_i \vec{v}_i \cdot \vec{\nabla} \left[\frac{1}{2} \rho_i \left(u_i^2 + v_i^2\right)\right] = - h_i \vec{v}_i \cdot \vec{\nabla} p_i . \tag{1.6}$$

Next multiply (1.4) by $\frac{1}{2} \rho_i \left(u_i^2 + v_i^2\right)$ to yield

$$\frac{1}{2} \rho_i \left(u_i^2 + v_i^2\right) \frac{\partial h_i}{\partial t} + \frac{1}{2} \rho_i \left(u_i^2 + v_i^2\right) \vec{\nabla} \cdot \vec{v}_i h_i = 0 . \tag{1.7}$$

Add (1.6) to (1.7) to obtain

$$\frac{\partial K_i}{\partial t} + \vec{\nabla}\cdot\vec{v}_i K_i = - h_i \vec{v}_i \cdot \vec{\nabla} p_i \tag{1.8}$$

or

$$\frac{\partial K_i}{\partial t} = \underset{(1)}{- \vec{\nabla}\cdot\vec{v}_i K_i} \underset{(2)}{- \vec{\nabla}\cdot\vec{v}_i h_i p_i} + \underset{(3)}{p_i \vec{\nabla}\cdot\vec{v}_i h_i} \ . \tag{1.9}$$

Term 1 is the layer-integrated flux of kinetic energy or the layer-integrated kinetic energy flux density. Term 2 is the layer-integrated pressure work term. Sometimes terms 1 and 2 are combined to form the layer-integrated energy flux divergence. If the layer-integrated energy flux divergence is integrated over the horizontal area, it represents the flux of energy through the side walls. Term 3 is discussed later.

It may appear that (1.9) contains a dependence on the depth, z, which (1.8) does not. It is readily shown that (1.9) does not depend on z.

From (A.7)

$$p_i(x,y,z,t) = \hat{p}_i(x,y,t) - \rho_i g z \ . \tag{1.10}$$

Substituting (1.10) into (1.9) gives

$$\frac{\partial K_i}{\partial t} = - \vec{\nabla}\cdot\vec{v}_i K_i - \vec{\nabla}\cdot\vec{v}_i h_i \hat{p}_i + \hat{p}_i \vec{\nabla}\cdot\vec{v}_i h_i \ , \tag{1.11}$$

which has no z dependence. Therefore, it does not matter whether $p_i$ or $\hat{p}_i$ is used in the energy calculations; there is no z dependence in $K_i$. However, $\hat{p}_i$ will be used in the remainder of the report to simplify the calculations.

The potential energy density in layer i is defined as $\rho_i g z$. The potential energy per unit area in a layer is defined as

$$P_i \equiv \int_{\tilde{z}_i}^{\tilde{z}_{i-1}} \rho_i g z \, dz \tag{1.12}$$

or

$$P_i = \frac{1}{2}\rho_i g\left[\left(\tilde{z}_{i-1}\right)^2 - \left(\tilde{z}_i\right)^2\right] \tag{1.13}$$

where $\tilde{z}_i$ is the depth at the bottom of layer i. Note that $\tilde{z}_i<0$ for $i\geq 1$.

From Figure 1, $\tilde{z}_i$ can be written as

$$\tilde{z}_i = \eta_{i+1} - \sum_{j=1}^{i} H_j \qquad \text{for } i\geq 1 \tag{1.14}$$

and

$$\tilde{z}_o = \eta_1 \ . \tag{1.15}$$

For fluids most of the potential energy is not available for conversion to other forms of energy. The system shown in Figure 1 is stably stratified; the density in a given layer is less than in the layer below. Therefore, the minimum potential energy, $\overline{P_i}$, occurs when the $\eta_i$'s are all zero or, from (1.14) and (1.15)

$$\overline{P_i} = \frac{1}{2}\rho_i g\left[\langle\tilde{z}_{i-1}\rangle^2 - \langle\tilde{z}_i\rangle^2\right] \tag{1.16}$$

where

$$\langle\tilde{z}_i\rangle = -\sum_{j=1}^{i} H_j \qquad \text{for } i\geq 1 \tag{1.17}$$

and

$$\langle\tilde{z}_o\rangle = 0 \ . \tag{1.18}$$

Note that both $P_i$ and $\overline{P_i}$ are less than zero. This result occurs because $z\equiv 0$ at the top, not at the bottom of the fluid. It is not convenient to set $z=0$ at the bottom of the fluid because the bottom layer is infinitely deep for the reduced-gravity model to be considered in section B.5.

The amount of potential energy that is available to be converted to kinetic energy is the difference between the actual potential energy and the minimum potential energy. Therefore, the available potential energy per unit area in a layer is

$$A_i \equiv P_i - \overline{P_i} \tag{1.19}$$

Substituting (1.13) and (1.16) into (1.19) gives

$$A_i = \left(\frac{\rho_i g}{2}\right) \left\{\left[\left(\tilde{z}_{i-1}\right)^2 - \{\tilde{z}_{i-1}\}^2\right] - \left[\left(\tilde{z}_i\right)^2 - \{\tilde{z}_i\}^2\right]\right\} . \tag{1.20}$$

Using (1.14), (1.15), (1.17) and (1.18) in (1.20) yields

$$A_i = \left(\frac{\rho_i g}{2}\right) \left\{\left[\left(\eta_i - \sum_{j=1}^{i-1} H_j\right)^2 - \left(-\sum_{j=1}^{i-1} H_j\right)^2\right] - \left[\left(\eta_{i+1} - \sum_{j=1}^{i} H_j\right)^2 - \left(-\sum_{j=1}^{i} H_j\right)^2\right]\right\} \qquad i>1 \tag{1.21}$$

or

$$A_i = \left(\frac{\rho_i g}{2}\right) \left[\left(\eta_i^2 - 2\eta_i \sum_{j=1}^{i-1} H_j\right) - \left(\eta_{i+1}^2 - 2\eta_{i+1} \sum_{j=1}^{i} H_j\right)\right] \qquad i>1 . \tag{1.22}$$

For the top layer, i=1, (1.22) becomes

$$A_1 = \left(\frac{\rho_1 g}{2}\right) \left[\eta_1^2 - \left(\eta_2^2 - 2\eta_2 H_1\right)\right] \qquad i=1 . \tag{1.23}$$

The change of available potential energy w.r.t. time is found by taking $\partial/\partial t$ of (1.22) and (1.23) to get

$$\frac{\partial A_i}{\partial t} = \rho_i g \left[\left(\eta_i - \sum_{j=1}^{i-1} H_j\right) \frac{\partial \eta_i}{\partial t} - \left(\eta_{i+1} - \sum_{j=1}^{i} H_j\right) \frac{\partial \eta_{i+1}}{\partial t}\right] \qquad i>1 \tag{1.24}$$

and

$$\frac{\partial A_1}{\partial t} = \rho_1 g \left[\eta_1 \frac{\partial \eta_1}{\partial t} - \left(\eta_2 - H_1\right)\frac{\partial \eta_2}{\partial t}\right] \qquad i=1 . \tag{1.25}$$

The total energy per unit area in a layer is defined as the sum of the kinetic energy per unit area and the available potential energy per unit area or $K_i+A_i$. Adding (1.11) and (1.24) gives the change w.r.t. time of the total energy per unit area in a layer as

$$\frac{\partial}{\partial t}\left(K_i + A_i\right) = -\vec{\nabla}\cdot\vec{v}_i\left(K_i + h_i\hat{p}_i\right) + \hat{p}_i\vec{\nabla}\cdot\vec{v}_i h_i$$

$$+ \rho_i g\left[\left(\eta_i - \sum_{j=1}^{i-1} H_j\right)\frac{\partial \eta_i}{\partial t} - \left(\eta_{i+1} - \sum_{j=1}^{i} H_j\right)\frac{\partial \eta_{i+1}}{\partial t}\right]. \qquad (1.26)$$

The first term on the RHS of (1.26) represents the horizontal transport of energy into the area. There are no external sources and sinks of energy for the finite depth model described in the Appendix. Therefore, the remaining terms on the RHS of (1.26) must represent vertical transport of energy or the exchange of energy between layers in the system. However, it is not immediately obvious from (1.26) which terms represent transfers between layers. Furthermore, transfers between kinetic and available potential energies within a layer should be possible, but these transfers are not obvious from (1.11) and (1.24). The remaining sections in Part B examine various ways to formulate energy transfers between layers and within a layer for both the finite depth and the reduced-gravity models.

Part B, Section 2

In this section the specific equations for energy transfer between layers and within a layer are obtained in a layer-layer formulation for a multilayer model.

From section B.1, (1.11) and (1.24), the equations for the local change of kinetic energy and of available potential energy with w.r.t. are

$$\frac{\partial K_i}{\partial t} = - \vec{\nabla}\cdot\vec{v}_i \left(K_i + h_i\hat{p}_i\right) + \hat{p}_i\vec{\nabla}\cdot\vec{v}_i h_i \qquad (2.1)$$

and

$$\frac{\partial A_i}{\partial t} = \rho_i g\left[\left(\eta_i - \sum_{j=1}^{i-1} H_j\right)\frac{\partial \eta_i}{\partial t} - \left(\eta_{i+1} - \sum_{j=1}^{i} H_j\right)\frac{\partial \eta_{i+1}}{\partial t}\right] . \qquad (2.2)$$

Figure 1 shows that the relation of $h_i$ to $\eta_i$ and $H_i$ is

$$h_i = \eta_i + H_i - \eta_{i+1} \qquad 1\leq i<N , \qquad (2.3)$$

while

$$h_N = \eta_N + H_N \qquad i=N . \qquad (2.4)$$

Substituting (1.10) into (1.5) gives

$$\hat{p}_i = \rho_i g\eta_1 - \sum_{j=1}^{i} \left(\rho_i - \rho_j\right) g h_j , \qquad (2.5)$$

while (1.4) is

$$\frac{\partial h_i}{\partial t} = - \vec{\nabla}\cdot\left(\vec{v}_i h_i\right) . \qquad (2.6)$$

The preceding set of equations, (2.1) - (2.6), can be reduced by eliminating the $\eta$'s. Sum (2.3) over all N-1 layers and add (2.4) to obtain

$$\sum_{j=1}^{N} h_j = \eta_1 + \sum_{j=1}^{N} H_j \qquad (2.7)$$

or

$$\eta_1 = \sum_{j=1}^{N} \left(h_j - H_j\right) . \tag{2.8}$$

If (2.3) is summed from layer 1 to layer i where i<N, the results are

$$\sum_{j=1}^{i} h_j = \eta_1 - \eta_{i+1} + \sum_{j=1}^{i} H_j . \tag{2.9}$$

Eliminate $\eta_1$ by using (2.8) in (2.9) to find

$$\eta_{i+1} = \sum_{j=1}^{N} \left(h_j - H_j\right) + \sum_{j=1}^{i} \left(H_j - h_j\right) = \sum_{j=i+1}^{N} \left(h_j - H_j\right) . \tag{2.10}$$

Use (2.10) in (2.3) to obtain

$$\eta_i = \sum_{j=i}^{N} \left(h_j - H_j\right) . \tag{2.11}$$

Substituting (2.8) in (2.5) gives

$$\hat{p}_i = \rho_i g \sum_{j=i}^{N} h_j - \rho_i g \sum_{j=1}^{N} H_j + \sum_{j=1}^{i-1} \rho_j g h_j . \tag{2.12}$$

Use (2.11) and (2.10) in (2.2) to get

$$\frac{\partial A_i}{\partial t} = \rho_i g \left[ \left( \sum_{j=i}^{N} \left(h_j - H_j\right) - \sum_{j=1}^{i-1} H_j \right) \frac{\partial}{\partial t} \left( \sum_{j=i}^{N} h_j \right) \right.$$

$$\left. - \left( \sum_{j=i+1}^{N} \left(h_j - H_j\right) - \sum_{j=i}^{i} H_j \right) \frac{\partial}{\partial t} \left( \sum_{j=i+1}^{N} h_j \right) \right] \tag{2.13}$$

or

$$\frac{\partial A_i}{\partial t} = \rho_i g \left[ \left( \sum_{j=i}^{N} h_j - \sum_{j=1}^{N} H_j \right) \frac{\partial}{\partial t} \left( \sum_{j=i}^{N} h_j \right) \right.$$

$$\left. - \left( \sum_{j=i+1}^{N} h_j - \sum_{j=1}^{N} H_j \right) \frac{\partial}{\partial t} \left( \sum_{j=i+1}^{N} h_j \right) \right] . \qquad (2.14)$$

The terms in 2.14 may be rearranged to yield

$$\frac{\partial A_i}{\partial t} = \rho_i g \left[ \left( \sum_{j=i+1}^{N} h_j - \sum_{j=1}^{N} H_j \right) \frac{\partial h_i}{\partial t} + h_i \sum_{j=i}^{N} \frac{\partial h_j}{\partial t} \right] . \qquad (2.15)$$

All terms on the RHS of (2.15) involve time derivatives of $h_i$. Multiply (2.6) by $\hat{p}_i$ and substitute in (2.1) to find

$$\frac{\partial K_i}{\partial t} = - \vec{\nabla} \cdot \vec{v}_i \left( K_i + h_i \hat{p}_i \right) - \hat{p}_i \frac{\partial h_i}{\partial t} . \qquad (2.16)$$

Add and subtract the term $\hat{p}_i \frac{\partial h_i}{\partial t}$ from (2.15) to find

$$\frac{\partial A_i}{\partial t} = \hat{p}_i \frac{\partial h_i}{\partial t} - \hat{p}_i \frac{\partial h_i}{\partial t} + \rho_i g \left[ \left( \sum_{j=i+1}^{N} h_j - \sum_{j=1}^{N} H_j \right) \frac{\partial h_i}{\partial t} + h_i \sum_{j=i}^{N} \frac{\partial h_j}{\partial t} \right] \qquad (2.17)$$

Replace the second $\hat{p}_i$ on the RHS of (2.17) with (2.12) to yield

$$\frac{\partial A_i}{\partial t} = \hat{p}_i \frac{\partial h_i}{\partial t} - \left[ \rho_i g \left( \sum_{j=i}^{N} h_j - \sum_{j=1}^{N} H_j \right) + \sum_{j=i}^{i-1} \rho_j g h_j \right] \frac{\partial h_i}{\partial t}$$

$$+ \rho_i g \left[ \left( \sum_{j=i+1}^{N} h_j - \sum_{j=1}^{N} H_j \right) \frac{\partial h_i}{\partial t} + h_i \sum_{j=i}^{N} \frac{\partial h_j}{\partial t} \right] \qquad (2.18)$$

or

$$\frac{\partial A_i}{\partial t} = \hat{p}_i \frac{\partial h_i}{\partial t} + \rho_i g h_i \sum_{j=i+1}^{N} \frac{\partial h_j}{\partial t} - \left( \sum_{j=i}^{i-1} \rho_j g h_j \right) \frac{\partial h_i}{\partial t} . \qquad (2.19)$$

Add and subtract $\left( \sum_{j=1}^{i-1} \rho_j g h_j \right) \left( \sum_{j=i+1}^{N} \frac{\partial h_j}{\partial t} \right)$ to obtain

$$\frac{\partial A_i}{\partial t} = \hat{p}_i \frac{\partial h_i}{\partial t} + \left( \sum_{j=1}^{i} \rho_j g h_j \right) \left( \sum_{j=i+1}^{N} \frac{\partial h_j}{\partial t} \right) - \left( \sum_{j=1}^{i-1} \rho_j g h_j \right) \left( \sum_{j=i}^{N} \frac{\partial h_j}{\partial t} \right) . \qquad (2.20)$$

The purpose of the preceding mathematical manipulations is now apparent. Both the kinetic energy and available potential energy relations have been written in terms $h_i$ and time derivatives of $h_i$. Furthermore, the various terms have been grouped to show the transfers of energy between layers. This grouping of terms for a given hydrodynamic model is usually neither obvious nor unique.

The energy transformations are more apparent when the relations for kinetic energy, (2.16), and available potential energy, (2.20), are written for the layers above, i-1, and below, i+1, the layer i as follows:

$$\frac{\partial K_{i-1}}{\partial t} = - \vec{\nabla} \cdot \vec{v}_{i-1} \left( K_{i-1} + h_{i-1} \hat{p}_{i-1} \right) - \hat{p}_{i-1} \frac{\partial h_{i-1}}{\partial t} \qquad (2.21)$$

(1) (2)

$$\frac{\partial A_{i-1}}{\partial t} = \hat{p}_{i-1} \frac{\partial h_{i-1}}{\partial t} + \left( \sum_{j=1}^{i-1} \rho_j g h_j \right) \left( \sum_{j=i}^{N} \frac{\partial h_j}{\partial t} \right) - \left( \sum_{j=1}^{i-2} \rho_j g h_j \right) \left( \sum_{j=i-1}^{N} \frac{\partial h_j}{\partial t} \right) \qquad (2.22)$$

(2) (3) (4)

$$\frac{\partial K_i}{\partial t} = - \vec{\nabla} \cdot \vec{v}_i \left( K_i + h_i \hat{p}_i \right) - \hat{p}_i \frac{\partial h_i}{\partial t} \qquad (2.23)$$

(5) (6)

$$\frac{\partial A_i}{\partial t} = \hat{p}_i \frac{\partial h_i}{\partial t} + \left( \sum_{j=1}^{i} \rho_j g h_j \right) \left( \sum_{j=i+1}^{N} \frac{\partial h_j}{\partial t} \right) - \left( \sum_{j=1}^{i-1} \rho_j g h_j \right) \left( \sum_{j=i}^{N} \frac{\partial h_j}{\partial t} \right) \qquad (2.24)$$

(6) (7) (3)

$$\frac{\partial K_{i+1}}{\partial t} = - \vec{\nabla} \cdot \vec{v}_{i+1} \left( K_{i+1} + h_{i+1} \hat{p}_{i+1} \right) - \hat{p}_{i+1} \frac{\partial h_{i+1}}{\partial t} \qquad (2.25)$$

(8) (9)

$$\frac{\partial A_{i+1}}{\partial t} = \hat{p}_{i+1} \frac{\partial h_{i+1}}{\partial t} + \left( \sum_{j=1}^{i+1} \rho_j g h_j \right) \left( \sum_{j=i+2}^{N} \frac{\partial h_j}{\partial t} \right)$$

(9) (10)

$$- \left( \sum_{j=1}^{i} \rho_j g h_j \right) \left( \sum_{j=i+1}^{N} \frac{\partial h_j}{\partial t} \right) \qquad (2.26)$$

(7)

Consider the energy transformations for layer i. Term 6 is the transfer between kinetic energy and available potential energy within the layer. Term 7 in (2.24) and (2.26) is the transfer of available potential energy between layer i and layer i+1. Term 3 in (2.22) and (2.24) is the transfer of available potential energy between layer i-1 and layer i. It follows that term 4 in (2.22) is the transfer of available potential energy between layer i-1 and layer i-2 and that term 10 in (2.26) is the transfer of available potential energy between layer i+1 and layer i+2. For completeness, it should be noted that term 2 in (2.21) and (2.22) and term 9 in (2.25) and (2.26) represent the transfer between kinetic and available potential energy in layers i-1 and i+1, respectively. Finally, terms 1, 5, and 8 are the energy flux divergence terms for layers i-1, i, and i+1, respectively. These relations are shown in the block diagram in Figure 2. Note that only layer i has all transfers marked.

The preceding discussion indicates that the kinetic energy and available potential energy relations, (2.16) and (2.20), may be written and interpreted as follows:

$$\frac{\partial K_i}{\partial t} = - \vec{\nabla} \cdot \vec{v}_i \left( K_i + h_i \hat{p}_i \right) - \hat{p}_i \frac{\partial h_i}{\partial t} \qquad (2.27)$$

(1) (2)

$$\frac{\partial A_i}{\partial t} = \hat{p}_i \frac{\partial h_i}{\partial t} + \left( \sum_{j=1}^{i} \rho_j g h_j \right) \left( \sum_{j=i+1}^{N} \frac{\partial h_j}{\partial t} \right) - \left( \sum_{j=1}^{i-1} \rho_j g h_j \right) \left( \sum_{j=i}^{N} \frac{\partial h_j}{\partial t} \right) \qquad (2.28)$$

(2) (3) (4)

Term 1 is the energy flux divergence for layer i. Term 2 is the transfer between kinetic energy and available potential energy in layer i. Term 3 is the transfer of available potential energy between layer i and layer i+1 while term 4 is the transfer between layer i and layer i-1.

The available potential energy relation, (2.28), shows transfers of energy with layers directly above and below the layer i. What happens when layer i is the top or bottom layer? For the top layer, i=1, (2.28) takes the form

$$\frac{\partial A_1}{\partial t} = \hat{p}_1 \frac{\partial h_1}{\partial t} + \rho_1 g h_1 \sum_{j=2}^{N} \frac{\partial h_j}{\partial t} \qquad (2.29)$$

The second term on the RHS of (2.29) is the transfer of available potential energy between layers 1 and 2. As there is no layer above layer 1, there is no term for the transfer of available potential energy between layer 1 and the layer above layer 1. For the bottom layer, i=N, (2.28) takes the form

$$\frac{\partial A_N}{\partial t} = \hat{p}_N \frac{\partial h_N}{\partial t} - \left( \sum_{j=1}^{N-1} \rho_j g h_j \right) \frac{\partial h_N}{\partial t} . \tag{2.30}$$

The second term on the RHS of (2.30) is the transfer of available potential energy between the bottom layer N and the layer above, N-1. As there is no layer below the bottom layer, there is no term for the transfer of available potential energy between layer N and the layer below layer N.

The total energy per unit area in a layer is $K_i+A_i$. Adding (2.27) and (2.28) gives the local time derivative of the total energy per unit area in a layer as

$$\frac{\partial}{\partial t}\left(K_i + A_i\right) = \underbrace{-\vec{\nabla}\cdot\vec{v}_i \left(K_i + h_i \hat{p}_i\right)}_{(1)} + \underbrace{\left( \sum_{j=1}^{i} \rho_j g h_j \right)\left( \sum_{j=i+1}^{N} \frac{\partial h_j}{\partial t} \right)}_{(3)} - \underbrace{\left( \sum_{j=1}^{i-1} \rho_j g h_j \right)\left( \sum_{j=i}^{N} \frac{\partial h_j}{\partial t} \right)}_{(4)} . \tag{2.31}$$

Here, term 1 is the energy flux divergence while terms 3 and 4 represent the exchange of total energy per unit area between the layer i and the layers below and above, respectively.

If the total energy per unit area in a layer, $K_i+A_i$, is summed over all layers, the sum is called the total energy per unit area of the system. Summing (2.31) over all N layers gives the local time derivative of the total energy per unit area for the system as

$$\sum_{i=1}^{N} \frac{\partial}{\partial t}\left(K_i + A_i\right) = - \sum_{i=1}^{N} \vec{\nabla}\cdot\vec{v}_i \left(K_i + h_i \hat{p}_i\right) . \tag{2.32}$$

Only flux divergence terms appear on the RHS of (2.32); there is no longer any transfer of energy between layers. Now integrate (2.32) over the horizontal area of the system to get the local time derivative of the total energy in the entire system:

$$\iint \sum_{i=1}^{N} \frac{\partial}{\partial t}\left(K_i + A_i\right) dxdy = - \iint \sum_{i=1}^{N} \vec{\nabla}\cdot\vec{v}_i \left(K_i + \hat{p}_i h_i\right) dxdy$$

$$= - \sum_{i=1}^{N} \int_{S_i} \left( K_i/h_i + \hat{p}_i \right) \vec{v}_i \cdot \hat{n}_i \, dS_i \quad (2.33)$$

where $S_i$ is the total surface of the side walls in layer i, $\hat{n}_i$ is the unit vector normal to the side walls in layer i, and $dS_i$ represents the differential area of the side walls in layer i. If the side walls are considered to be solid, there can be no flux of energy through them so (2.33) becomes

$$\iint \sum_{i=1}^{N} \frac{\partial}{\partial t} \left( K_i + A_i \right) dxdy = \frac{\partial}{\partial t} \iint \sum_{i=1}^{N} \left( K_i + A_i \right) dxdy = 0 \, . \quad (2.34)$$

This relation merely states that the total energy is conserved for an inviscid multilayer hydrodynamic model with no external sources or sinks.

The energy specifications and relations in the layer-layer formulation for the finite depth model are given for the one-, two-, and three-layer cases in sections C.8, C.9, and C.10, respectively.

For some studies the total kinetic energy and the total available potential energy are treated separately. The local time derivative of the total kinetic energy for the system is found by summing (2.27) over all N layers and then integrating over the horizontal area:

$$\iint \sum_{i=1}^{N} \frac{\partial K_i}{\partial t} dxdy = - \iint \sum_{i=1}^{N} \vec{\nabla} \cdot \vec{v} \left( K_i + h_i \hat{p}_i \right) dxdy$$

$$- \iint \sum_{i=1}^{N} \hat{p}_i \frac{\partial h_i}{\partial t} dxdy \quad (2.35)$$

or, for solid side walls,

$$\iint \sum_{i=1}^{N} \frac{\partial K_i}{\partial t} dxdy = - \iint \sum_{i=1}^{N} \hat{p}_i \frac{\partial h_i}{\partial t} dxdy \quad (2.36)$$

Similarly the local time derivative of the total available potential energy for the system is found by summing (2.28) over all N layers and then integrating over the horizontal area:

$$\iint \sum_{i=1}^{N} \frac{\partial A_i}{\partial t} dxdy = \iint \sum_{i=1}^{N} \hat{p}_i \frac{\partial h_i}{\partial t} dxdy \quad (2.37)$$

If (2.36) and (2.37) are added, the result is (2.34) which shows that energy is transferred between kinetic and available potential forms even though the total energy for the basin is conserved. See Semtner and Mintz (1977) and Robinson et al. (1979) for further examples of basin-integrated energetics.

Part B, Section 3

In sections B.1 and B.2 the available potential energy per layer has been used in determining energy transfers. However, there is another way to treat the available potential energy that will be derived in this section.

From (1.22), the available potential energy per unit area, $A_i$, in layer i can be written as

$$A_i = \frac{1}{2}\rho_i g\left[\left(\eta_i^2 - 2\eta_i \sum_{j=1}^{i-1} H_j\right) - \left(\eta_{i+1}^2 - 2\eta_{i+1}\sum_{j=1}^{i} H_j\right)\right] . \tag{3.1}$$

The total available potential energy per unit area for the entire system is found by summing (3.1) over the N layers to get

$$\sum_{i=1}^{N} A_i = \sum_{i=1}^{N} \frac{1}{2}\rho_i g\left[\left(\eta_i^2 - 2\eta_i \sum_{j=1}^{i-1} H_j\right) - \left(\eta_{i+1}^2 - 2\eta_{i+1}\sum_{j=1}^{i} H_j\right)\right] \tag{3.2}$$

or

$$\sum_{i=1}^{N} A_i = \sum_{i=1}^{N} \left(\frac{1}{2}(\rho_i - \rho_{i-1})g\right)\left(\eta_i^2 - 2\eta_i \sum_{j=1}^{i-1} H_j\right) . \tag{3.3}$$

Examination of (3.3) reveals that available potential energy is associated with each interface deviation, $\eta_i$. Define the available potential energy associated with interface deviation, $\eta_i$, such that

$$\tilde{A}_i \equiv \left(\frac{1}{2}(\rho_i - \rho_{i-1})g\right)\left(\eta_i^2 - 2\eta_i \sum_{j=1}^{i-1} H_j\right) \qquad i>1 \tag{3.4}$$

and

$$\tilde{A}_1 \equiv \frac{1}{2}\rho_1 g\,\eta_1^2 \qquad i=1 . \tag{3.5}$$

If (3.4) is summed over N layers, the result is

$$\sum_{i=1}^{N} \tilde{A}_i = \sum_{i=1}^{N} \left(\frac{1}{2}(\rho_i - \rho_{i-1})g\right)\left(\eta_i^2 - 2\eta_i \sum_{j=1}^{i-1} H_j\right) \tag{3.6}$$

Comparison of (3.3) and (3.6) shows that the total available potential energy per unit area in the system is the same no matter that formulation of available potential energy is used: layer, $A_i$, or interface, $\tilde{A}_i$.

Take $\partial/\partial t$ of (3.4) and (3.5) to get

$$\frac{\partial \tilde{A}_i}{\partial t} = \left(\rho_i - \rho_{i-1}\right) g \left(\eta_i - \sum_{j=1}^{i-1} H_j\right) \frac{\partial \eta_i}{\partial t} \qquad i>1 \qquad (3.7)$$

and

$$\frac{\partial \tilde{A}_1}{\partial t} = \rho_1 g\, \eta_1 \frac{\partial \eta_1}{\partial t} \qquad i=1 \qquad (3.8)$$

From (2.16), the change w.r.t. time of kinetic energy per unit area in a layer is

$$\frac{\partial K_i}{\partial t} = - \vec{\nabla}\cdot\vec{v}_i \left(K_i + h_i \hat{p}_i\right) - \hat{p}_i \frac{\partial h_i}{\partial t} \qquad (3.9)$$

where, from (2.3) and (2.5),

$$h_i = \eta_i + H_i - \eta_{i+1} \qquad (3.10)$$

and

$$\hat{p}_i = \rho_i g\, \eta_1 - \sum_{j=1}^{i} \left(\rho_i - \rho_j\right) g\, h_j \qquad (3.11)$$

The interface available potential energy, $\tilde{A}_i$, is given in (3.4) - (3.8) in terms of the interface deviation, $\eta_i$, so use (3.10) in the last term on the RHS of (3.9) to eliminate $h_i$. The result is

$$\frac{\partial K_i}{\partial t} = - \vec{\nabla}\cdot\vec{v}_i \left(K_i + h_i \hat{p}_i\right) - \hat{p}_i \left(\frac{\partial \eta_i}{\partial t} - \frac{\partial \eta_{i+1}}{\partial t}\right) . \qquad (3.12)$$

Now evaluate $\hat{p}_i - \hat{p}_{i-1}$ by using (3.11) twice to get

$$\hat{p}_i - \hat{p}_{i-1} = \rho_i g\, \eta_1 - \sum_{j=1}^{i} \left(\rho_i - \rho_j\right) g\, h_j - \rho_{i-1} g\, \eta_1 + \sum_{j=1}^{i-1} \left(\rho_{i-1} - \rho_j\right) g\, h_j \qquad (3.13)$$

or

$$\hat{p}_i - \hat{p}_{i-1} = \left(\rho_i - \rho_{i-1}\right) g \left[\eta_1 - \sum_{j=1}^{i-1} h_j\right] . \tag{3.14}$$

Sum (3.10) from layer 1 to layer i-1 to get

$$\sum_{j=1}^{i-1} h_j = \sum_{j=1}^{i-1} \left(\eta_i - \eta_{i+1}\right) + \sum_{j=1}^{i-1} H_j \tag{3.15}$$

or

$$\sum_{j=1}^{i-1} h_j = \left(\eta_1 - \eta_i\right) + \sum_{j=1}^{i-1} H_j . \tag{3.16}$$

Use (3.16) in (3.14) to find

$$\hat{p}_i - \hat{p}_{i-1} = \left(\rho_i - \rho_{i-1}\right) g \left[\eta_1 - \left(\eta_1 - \eta_i\right) - \sum_{j=1}^{i-1} H_j\right] \tag{3.17}$$

or

$$\hat{p}_i - \hat{p}_{i-1} = \left(\rho_i - \rho_{i-1}\right) g \left[\eta_i - \sum_{j=1}^{i-1} H_j\right] . \tag{3.18}$$

Use (3.18) in (3.7) to obtain

$$\frac{\partial \tilde{A}_i}{\partial t} = \left(\hat{p}_i - \hat{p}_{i-1}\right) \frac{\partial \eta_i}{\partial t} . \tag{3.19}$$

Both kinetic energy and interface available potential energy relations have been written in terms of the interface deviation, $\eta_i$, and its derivative w.r.t. time. It is useful to consider the energy relations (3.19) and (3.12) for adjacent interfaces and layers. For i-1, i, and i+1, (3.19) and (3.12) become

$$\frac{\partial \tilde{A}_{i-1}}{\partial t} = \underset{(1)}{\hat{p}_{i-1} \frac{\partial \eta_{i-1}}{\partial t}} - \underset{(2)}{\hat{p}_{i-2} \frac{\partial \eta_{i-1}}{\partial t}} , \tag{3.20}$$

$$\frac{\partial K_{i-1}}{\partial t} = - \vec{\nabla}\cdot\vec{v}_{i-1}\left(K_{i-1} + h_{i-1}\hat{p}_{i-1}\right) - \hat{p}_{i-1}\frac{\partial \eta_{i-1}}{\partial t} + \hat{p}_{i-1}\frac{\partial \eta_i}{\partial t}, \qquad (3.21)$$

(3) (1) (4)

$$\frac{\partial \tilde{A}_i}{\partial t} = \hat{p}_i\frac{\partial \eta_i}{\partial t} - \hat{p}_{i-1}\frac{\partial \eta_i}{\partial t}, \qquad (3.22)$$

(5) (4)

$$\frac{\partial K_i}{\partial t} = - \vec{\nabla}\cdot\vec{v}_i\left(K_i + h_i\hat{p}_i\right) - \hat{p}_i\frac{\partial \eta_i}{\partial t} + \hat{p}_i\frac{\partial \eta_{i+1}}{\partial t}, \qquad (3.23)$$

(6) (5) (7)

$$\frac{\partial \tilde{A}_{i+1}}{\partial t} = \hat{p}_{i+1}\frac{\partial \eta_{i+1}}{\partial t} - \hat{p}_i\frac{\partial \eta_{i+1}}{\partial t}, \qquad (3.24)$$

(8) (7)

$$\frac{\partial K_{i+1}}{\partial t} = - \vec{\nabla}\cdot\vec{v}_{i+1}\left(K_{i+1} + h_{i+1}\hat{p}_{i+1}\right) - \hat{p}_{i+1}\frac{\partial \eta_{i+1}}{\partial t} + \hat{p}_{i+1}\frac{\partial \eta_{i+2}}{\partial t}. \qquad (3.25)$$

(9) (8) (10)

The transfer between kinetic energy in a given layer and the available potential energy for the interface above the given layer is given by terms 1, 5, and 8 for layers i-1, i, and i+1, respectively. Similarly, the transfer between kinetic energy in a given layer and the available potential energy for the interface below the given layer is given by terms 2, 4, 7, and 10 for layers i-2, i-1, i, and i+1, respectively. Terms 3, 6, and 9 are the energy flux divergences for layers i-1, i, and i+1, respectively. These relations are shown in the block diagram in Figure 3.

The preceding discussion indicates that kinetic energy and interface available potential energy relations may be written and interpreted as follows:

$$\frac{\partial K_i}{\partial t} = - \vec{\nabla}\cdot\vec{v}_i\left(K_i + h_i\hat{p}_i\right) - \hat{p}_i\frac{\partial \eta_i}{\partial t} + \hat{p}_i\frac{\partial \eta_{i+1}}{\partial t} \qquad 1\leq i<N \qquad (3.26)$$

(1) (2) (3)

$$\frac{\partial \tilde{A}_i}{\partial t} = \hat{p}_i\frac{\partial \eta_i}{\partial t} - \hat{p}_{i-1}\frac{\partial \eta_i}{\partial t} \qquad 1<i\leq N \qquad (3.27)$$

(2) (4)

Terms 2 and 3 are the transfers between the kinetic energy in layer i and the available potential energy for the interfaces above and below layer i, respectively. Term 4 is the transfer between the available potential energy for interface i and the kinetic energy in layer i-1, the layer above interface i. Term 1 is the usual energy flux divergence associated with layer i. What happens to these relations for the top and bottom layers? For the top layer, (3.27) takes the form

$$\frac{\partial \tilde{A}_1}{\partial t} = \hat{p}_1 \frac{\partial \eta_1}{\partial t} \qquad i=1 \ . \tag{3.28}$$

As there is no layer above interface 1, no term exists for the transfer of energy to that layer. For the bottom layer, i=N, (3.26) becomes

$$\frac{\partial K_N}{\partial t} = - \vec{\nabla}\cdot\vec{v}_N \left(K_N + h_N \hat{p}_N\right) - \hat{p}_N \frac{\partial \eta_N}{\partial t} \qquad i=N \ . \tag{3.29}$$

As there is no interface below layer N, there is no term for the transfer of any energy associated with such an interface.

The local time derivative of the total kinetic energy per unit area is found by summing (3.26) over N-1 layers and adding (3.29) to get

$$\sum_{i=1}^{N} \frac{\partial K_i}{\partial t} = - \sum_{i=1}^{N} \vec{\nabla}\cdot\vec{v}_i \left(K_i + h_i \hat{p}_i\right) - \sum_{i=1}^{N} \left(\hat{p}_i - \hat{p}_{i-1}\right) \frac{\partial \eta_i}{\partial t} \ . \tag{3.30}$$

The local time derivative of the total available potential energy per unit area is found by summing (3.27) from i=2 to i=N and adding (3.28) to find

$$\sum_{i=1}^{N} \frac{\partial \tilde{A}_i}{\partial t} = \hat{p}_1 \frac{\partial \eta_1}{\partial t} + \sum_{i=2}^{N} \left(\hat{p}_i - \hat{p}_{i-1}\right) \frac{\partial \eta_i}{\partial t} \tag{3.31}$$

or, rearranging terms,

$$\sum_{i=1}^{N} \frac{\partial \tilde{A}_i}{\partial t} = \sum_{i=1}^{N} \left(\hat{p}_i - \hat{p}_{i-1}\right) \frac{\partial \eta_i}{\partial t} \ . \tag{3.32}$$

Adding (3.30) and (3.32) gives the local time derivative of the total energy per unit area as

$$\sum_{i=1}^{N} \frac{\partial}{\partial t} \left( K_i + \tilde{A}_i \right) = - \sum_{i=1}^{N} \vec{\nabla} \cdot \vec{v}_i \left( K_i + h_i \hat{p}_i \right) . \qquad (3.33)$$

Compare this energy relation to (2.32) that is the change in the total energy per unit area for the system in the layer-layer formulation. In both formulations only energy flux divergence terms contribute to a change w.r.t. time in the total energy per unit area.

If (3.33) is integrated over the horizontal area of the basin, the result is

$$\frac{\partial}{\partial t} \iint \sum_{i=1}^{N} \left( K_i + \tilde{A}_i \right) dxdy = - \sum_{i=1}^{N} \iint \vec{\nabla} \cdot \vec{v}_i \left( K_i + \hat{p}_i h_i \right) dxdy$$

$$= - \sum_{i=1}^{N} \int_{S_i} \left( K_i / h_i + \hat{p}_i \right) \vec{v}_i \cdot \hat{n}_i \, dS_i = 0 \qquad (3.34)$$

for solid side walls as found in (2.34) for the layer-layer formulation of energetics.

The energy specifications and relations in the layer-interface formulation for the finite depth model are given in sections D.13, D.14, and D.15 for the one-, two-, and three-layer cases, respectively.

Part B, Section 4

The major difference between the layer-layer and the layer-interface formulations in sections B.2 and B.3 lies in the grouping of terms. The total kinetic energy, summed over all layers, remains the same. The total available potential energy, whether summed over all layers in the layer formulation or summed over all interfaces in the interface formulation, remains the same for the system as shown in (3.3) and (3.6). The question naturally arises as to whether there are other ways of formulating the energy transfers. In this section, a third way of formulating energy transfers for the finite depth hydrodynamic model is derived. It is included here as an example of an energy formulation that is correct mathematically, but unsatisfactory in terms of the physical mechanisms of transfer represented.

Consider the layer-level formulation in section B.3. The relevant energy transfer relations, (3.26) and (3.27), are

$$\frac{\partial K_i}{\partial t} = - \vec{\nabla}\cdot\vec{v}_i \left(K_i + h_i \hat{p}_i\right) - \hat{p}_i \frac{\partial \eta_i}{\partial t} + \hat{p}_i \frac{\partial \eta_{i+1}}{\partial t} \tag{4.1}$$

and

$$\frac{\partial A_i}{\partial t} = \left(\hat{p}_i - \hat{p}_{i-1}\right) \frac{\partial \eta_i}{\partial t} \, . \tag{4.2}$$

Now add and subtract the term $\hat{p}_{i-1} \frac{\partial \eta_i}{\partial t}$ to (4.1) to derive

$$\frac{\partial K_i}{\partial t} = - \vec{\nabla}\cdot\vec{v}_i \left(K_i + h_i \hat{p}_i\right) - \left(\hat{p}_i - \hat{p}_{i-1}\right)\frac{\partial \eta_i}{\partial t} + \hat{p}_i \frac{\partial \eta_{i+1}}{\partial t} - \hat{p}_{i-1} \frac{\partial \eta_i}{\partial t} \, . \tag{4.3}$$

It is useful to consider (4.2) and (4.3) for adjacent interfaces and levels. For i-1, i, and i+1, these relations become

$$\frac{\partial \tilde{A}_{i-1}}{\partial t} = \underset{(1)}{\left(\hat{p}_{i-1} - \hat{p}_{i-2}\right)} \frac{\partial \eta_{i-1}}{\partial t} \, , \tag{4.4}$$

$$\frac{\partial K_{i-1}}{\partial t} = - \vec{\nabla}\cdot\vec{v}_{i-1} \underset{(2)}{\left(K_{i-1} + h_{i-1}\hat{p}_{i-1}\right)} - \underset{(1)}{\left(p_{i-1} - \hat{p}_{i-2}\right)} \frac{\partial \eta_{i-1}}{\partial t}$$

$$+ \underset{(3)}{\hat{p}_{i-1}} \frac{\partial \eta_i}{\partial t} - \underset{(4)}{\hat{p}_{i-2}} \frac{\partial \eta_{i-1}}{\partial t} , \tag{4.5}$$

$$\frac{\partial \tilde{A}_i}{\partial t} = \underset{(5)}{\left(\hat{p}_i - \hat{p}_{i-1}\right)} \frac{\partial \eta_i}{\partial t} , \tag{4.6}$$

$$\frac{\partial K_i}{\partial t} = - \vec{\nabla}\cdot\vec{v}_i \underset{(6)}{\left(K_i + h_i \hat{p}_i\right)} - \underset{(5)}{\left(\hat{p}_i - \hat{p}_{i-1}\right)} \frac{\partial \eta_i}{\partial t}$$

$$+ \underset{(7)}{\hat{p}_i} \frac{\partial \eta_{i+1}}{\partial t} - \underset{(3)}{\hat{p}_{i-1}} \frac{\partial \eta_i}{\partial t} , \tag{4.7}$$

$$\frac{\partial \tilde{A}_{i+1}}{\partial t} = \underset{(8)}{\left(\hat{p}_{i+1} - \hat{p}_i\right)} \frac{\partial \eta_{i+1}}{\partial t} , \tag{4.8}$$

$$\frac{\partial K_{i+1}}{\partial t} = - \vec{\nabla}\cdot\vec{v}_{i+1} \underset{(9)}{\left(K_{i+1} + h_{i+1}\hat{p}_{i+1}\right)} - \underset{(8)}{\left(\hat{p}_{i+1} - \hat{p}_i\right)} \frac{\partial \eta_{i+1}}{\partial t}$$

$$+ \underset{(10)}{\hat{p}_{i+1}} \frac{\partial \eta_{i+2}}{\partial t} - \underset{(7)}{\hat{p}_i} \frac{\partial \eta_{i+1}}{\partial t} . \tag{4.9}$$

Terms 1, 5, and 8 represent the transfer between the kinetic energy in a layer and the available potential energy associated with the interface directly above that layer for layers i-1, i, and i+1, respectively. Term 3 is the transfer of kinetic energy between layers i-1 and i and term 7 is the same transfer between layers i and i+1. Terms 4 and 10 are the transfer of kinetic energy between layers i-1 and i-2 and layers i+1 and i+2, respectively. Terms 2, 6, and 9 represent the energy flux divergence per unit area in layers i-1, i, and i+1, respectively. These relations are shown in the block diagram in Figure 4.

The preceding discussion indicates that the kinetic energy and available potential energy relations, (4.2) and (4.3) can be written and interpreted as follows:

$$\frac{\partial K_i}{\partial t} = - \vec{\nabla}\cdot\vec{v}_i \left(K_i + h_i \hat{p}_i\right) - \left(\hat{p}_i - \hat{p}_{i-1}\right)\frac{\partial \eta_i}{\partial t}$$

(1) (2)

$$+ \hat{p}_i \frac{\partial \eta_{i+1}}{\partial t} - \hat{p}_{i-1} \frac{\partial \eta_i}{\partial t} \qquad 1<i\leq N \tag{4.10}$$

(3) (4)

$$\frac{\partial \tilde{A}_i}{\partial t} = \left(\hat{p}_i - \hat{p}_{i-1}\right) \frac{\partial \eta_i}{\partial t} \qquad i>1 \tag{4.11}$$

(2)

Term 1 is the energy flux divergence for layer i. Term 2 is the transfer between kinetic energy in layer i and the available potential energy associated with the interface i above layer i. Term 3 is the exchange of kinetic energy between layer i and the layer above, i-1. Term 4 is the exchange of kinetic energy between layer i and the layer below, i+1.

The formulation of energy transfers in this section is unsatisfactory as the physical mechanisms depicted are unrealistic. As shown in Figure 4, kinetic energy in a layer is transferred to and from available potential energy for the interface above the layer. There is no reason such an exchange cannot take place between a given layer and the interface below, as shown in Figure 3, yet no mechanism for such a transfer occurs in the formulation for this section. Furthermore, this formulation allows for the transfer of kinetic energy between layers. However, the inviscid ocean model described in the Appendix does not permit the direct transfer of mass from one layer to another, so there is no physical mechanism to allow the direct exchange of kinetic energy between layers. These two shortcomings indicate that the formulation in this section, although mathematically correct, is not satisfactory in terms of physical mechanisms needed for this ocean model.

Part B, Section 5

In this section the specific equations for energy transfer between layers and within a layer are found in a layer-layer formulation for a multilayer reduced-gravity model. The reduced-gravity model contains one more assumption than is given in the Appendix: the bottom layer, N, is assumed to be of infinite depth. How does this additional assumption alter the energy relations found in section B.2?

From section B.1, (1.1), the kinetic energy per unit area in layer N, $K_N$, is written as

$$K_N = \frac{1}{2} h_N \rho_N \left(u_N^2 + v_N^2\right) , \tag{5.1}$$

where, from (2.4)

$$h_N = \eta_N + H_N . \tag{5.2}$$

Substitute (5.2) in (5.1) to derive

$$K_N = \frac{1}{2} \rho_N \left(\eta_N + H_N\right) \left(u_N^2 + v_N^2\right) . \tag{5.3}$$

Now as $H_N \to \infty$, $K_N \to \infty$, unless $u_N$ and $v_N \to 0$. It is not physically reasonable to have infinite kinetic energy so the velocity in the bottom layer is set to zero in a reduced-gravity model. It follows that $K_N = 0$ and remains zero, i.e. $\partial K_N / \partial t = 0$. Hence, the bottom layer is said to be inactive in a reduced-gravity model.

The starting equations, derived in the Appendix and listed in Table 1, are

$$\frac{\partial u_i}{\partial t} + u_i \left(\frac{\partial u_i}{\partial x}\right) + v_i \left(\frac{\partial u_i}{\partial x}\right) - f v_i = - \frac{1}{\rho_i} \left(\frac{\partial p_i}{\partial x}\right) , \tag{5.4}$$

$$\frac{\partial v_i}{\partial t} + u_i \left(\frac{\partial v_i}{\partial x}\right) + v_i \left(\frac{\partial v_i}{\partial x}\right) + f u_i = - \frac{1}{\rho_i} \left(\frac{\partial p_i}{\partial y}\right) , \tag{5.5}$$

$$\frac{\partial h_i}{\partial t} + u_i \left(\frac{\partial h_i}{\partial x}\right) + v_i \left(\frac{\partial h_i}{\partial y}\right) + h_i \left(\frac{\partial u_i}{\partial x}\right) + h_i \left(\frac{\partial v_i}{\partial y}\right) = 0 , \tag{5.6}$$

$$p_i = \rho_i g \left(\eta_1 - z\right) - \sum_{j=1}^{i} \left(\rho_i - \rho_j\right) g h_j . \tag{5.7}$$

These equations for layers i<N do not depend on the depth of layer N and so remain valid for layers i<N in the reduced-gravity model. What happens to these equations for the bottom layer, i=N, in the reduced-gravity model?

Consider first the pressure equation, (5.7). From (A.7), the pressure, $p_i$, can be written as

$$p_i(x,y,z,t) = \hat{p}_i(x,y,t) - \rho_i g z. \tag{5.8}$$

If (5.8) is used in (5.7) to eliminate $p_i$, the result for the bottom layer is

$$\hat{p}_N = \rho_N g\, \eta_1 - \sum_{j=1}^{N} \left(\rho_N - \rho_j\right) g\, h_j \ . \tag{5.9}$$

For the bottom layer $\rho_j = \rho_N$ so (5.9) may be written as

$$\hat{p}_N = \rho_N g\, \eta_1 - \sum_{j=1}^{N-1} \left(\rho_N - \rho_j\right) g\, h_j \tag{5.10}$$

Thus, $\hat{p}_N$ does not depend on the depth of the bottom layer and (5.10) may be used for the reduced-gravity model.

Consider next the momentum equations (5.4) - (5.5) for the bottom layer in the reduced-gravity model. The velocity is zero in the bottom layer so these equations become

$$0 = - \frac{\partial p_N}{\partial x} = - \frac{\partial \hat{p}_N}{\partial x} \tag{5.11}$$

and

$$0 = - \frac{\partial p_N}{\partial y} = - \frac{\partial \hat{p}_N}{\partial y} \ . \tag{5.12}$$

Thus, for the reduced-gravity model, the pressure in the bottom layer is not a function of x and y, so $\hat{p}_N$ is a function of time only. Next, average (5.10) over the area of the model:

$$\langle \hat{p}_N \rangle = \rho_N g\, \langle \eta_1 \rangle - \sum_{j=1}^{N-1} \left(\rho_N - \rho_j\right) g\, \langle h_j \rangle \tag{5.13}$$

where the angle brackets indicate an average over the basin area. From (2.3) the relation between $\eta_i$ and $h_i$ is given as

$$h_i = \eta_i - \eta_{i+1} + H_i \tag{5.14}$$

The basin area-average of (5.14) is

$$\langle h_i \rangle = \langle \eta_i \rangle - \langle \eta_{i+1} \rangle + H_i = H_i \qquad i<N \tag{5.15}$$

because $H_i$ is defined as the basin area-averaged thickness of layer i. Furthermore, $\eta_i$ is the difference between the actual and basin area-averaged position of the top of layer i. It follows that

$$\langle \eta_i \rangle = 0 \qquad 1 \leq i \leq N \ . \tag{5.16}$$

Now the mass in layer i, $M_i$, is given by

$$M_i = \rho_i \langle h_i \rangle \times \text{Area} = \rho_i H_i \times \text{Area} \qquad i<N \ . \tag{5.17}$$

For the multilayer hydrodynamic models considered here, mass in a layer is assumed to be conserved so $M_i$ and therefore $H_i$ do not vary with time. If (5.15) and (5.16) are used in (5.13), the result is

$$\langle \hat{p}_N \rangle = - \sum_{j=1}^{N-1} \left(\rho_N - \rho_j\right) g \, H_j \tag{5.18}$$

which is a constant for the system and not a function of x, y, or t. Note that (5.18) is correct for both the finite depth and the reduced-gravity models. However, for the reduced-gravity model, it follows from (5.11) and (5.12) that

$$\hat{p}_N = \langle \hat{p}_N \rangle \tag{5.19}$$

and from (5.10), (5.18), and (5.19),

$$\hat{p}_N = \rho_N g \, \eta_1 - \sum_{j=1}^{N-1} \left(\rho_N - \rho_j\right) g \, h_j = - \sum_{j=1}^{N-1} \left(\rho_N - \rho_j\right) g \, H_j \ . \tag{5.20}$$

Thus, in the reduced-gravity model $\hat{p}_N$ is also not a function of time. Sum (5.14) over N-1 layers to obtain

$$\eta_1 = \eta_N + \sum_{j=1}^{N-1} \left(h_j - H_j\right) . \tag{5.21}$$

Use (5.21) to eliminate $\eta_1$, in (5.20); the result is

$$\hat{p}_N = \rho_N g\, \eta_N + \sum_{j=1}^{N-1} g \left(\rho_j h_j - \rho_N H_j\right) = - \sum_{j=1}^{N-1} \left(\rho_N - \rho_j\right) g\, H_j . \tag{5.22}$$

Take $\partial/\partial t$ of (5.22) to find that

$$\frac{\partial \hat{p}_N}{\partial t} = \rho_N g \frac{\partial \eta_N}{\partial t} + \sum_{j=1}^{N-1} g\rho_j \frac{\partial h_j}{\partial t} = 0 . \tag{5.23}$$

The deriviative w.r.t. time of (5.2) is

$$\frac{\partial h_N}{\partial t} = \frac{\partial \eta_N}{\partial t} , \tag{5.24}$$

that is valid even if $H_N \to \infty$. So (5.23) can be written as

$$\frac{\partial \eta_N}{\partial t} = - \sum_{j=1}^{N-1} \left(\frac{\rho_j}{\rho_N}\right) \frac{\partial h_j}{\partial t} = \frac{\partial h_N}{\partial t} . \tag{5.25}$$

Note that (5.6) has been obtained in the Appendix by integrating over the layer depth. Thus, in a reduced-gravity model, (5.6) does not apply to the bottom layer, but (5.24) or (5.25) is used to evaluate the change w.r.t. time of the interface deviation, $\eta_N$.

The preceding discussion reveals that the assumption of infinite depth for the bottom layer in a multilayer hydrodynamic model has three direct consequences. First, the velocity in the bottom layer is zero. Second, the pressure in bottom layer is independent of x, y, and t. Third, as shown in (5.24), the interface deviation at the top of the bottom layer, $\eta_N$, cannot be specified independently, but is directly coupled to the sum of all other interface deviations.

Consider now the energy relations for a reduced-gravity, hydrodynamic model. For layers 1 to N-1, (5.4) - (5.7) are the same equations used to

derive the energy relations in section B.2 for the layer-layer formulation for the finite depth model. These energy relations are, from (2.27) and (2.28),

$$\frac{\partial K_i}{\partial t} = - \vec{\nabla}\cdot\vec{v}_i \left(K_i + h_i \hat{p}_i\right) - \hat{p}_i \frac{\partial h_i}{\partial t} \tag{5.26}$$

and

$$\frac{\partial A_i}{\partial t} = \hat{p}_i \frac{\partial h_i}{\partial t} + \left( \sum_{j=1}^{i} \rho_j g h_j \right) \left( \sum_{j=i+1}^{N} \frac{\partial h_j}{\partial t} \right) - \left( \sum_{j=1}^{i-1} \rho_j g h_j \right) \left( \sum_{j=i}^{N} \frac{\partial h_j}{\partial t} \right) . \tag{5.27}$$

For the bottom layer, the discussion following (5.3) has shown that

$$\frac{\partial K_N}{\partial t} = 0 . \tag{5.28}$$

The general expression for the local time derivative of available potential energy is derived in section B.1 and does not depend on the layer depth, only on interface deviation. Thus, this relation, (1.24), is valid for the reduced-gravity model and is, for layer N,

$$\frac{\partial A_N}{\partial t} = \rho_N g \left( \eta_N - \sum_{j=1}^{N-1} H_j \right) \frac{\partial \eta_N}{\partial t} . \tag{5.29}$$

From (5.22), $\eta_N$ can be written as

$$\eta_N = - \sum_{j=1}^{N-1} \left( \frac{\rho_j}{\rho_N} \right) \left( h_j - H_j \right) ; \tag{5.30}$$

If (5.14) is used in (5.30), the result is

$$\left( \rho_N - \rho_{N-1} \right) \eta_N = - \sum_{j=1}^{N-1} \left( \rho_j - \rho_{j-1} \right) \eta_j . \tag{5.31}$$

Therefore, in the reduced-gravity model, the interface deviation at the top of the bottom layer is determined completely by a sum of the other interface deviations multiplied by density differences. This result, (5.31), is a direct consequence of the fact that the pressure in the bottom layer is independent of horizontal position and time, and it does not hold for the finite depth model.

Use (5.30) and (5.24) in (5.29) to eliminate $\eta_N$; the result is

$$\frac{\partial A_N}{\partial t} = g \sum_{j=1}^{N-1} \left[ \left( \rho_j - \rho_N \right) H_j - \rho_j h_j \right] \frac{\partial h_N}{\partial t} \, . \tag{5.32}$$

With the aid of (5.22), (5.32) can be written as

$$\frac{\partial A_N}{\partial t} = \hat{p}_N \frac{\partial h_N}{\partial t} - \left( \sum_{j=1}^{N-1} \rho_j g h_j \right) \frac{\partial h_N}{\partial t} \, . \tag{5.33}$$

A mathematically consistent set of energy relations has been derived for a reduced-gravity model: (5.26) and (5.27) for layers 1 to N-1 and (5.28) and (5.33) for layer N. However, there is a problem with the interpretation of the energy terms. To illustrate this problem, note that for i=N-1, (5.27) becomes

$$\frac{\partial A_{N-1}}{\partial t} = \hat{p}_{N-1} \frac{\partial h_{N-1}}{\partial t} + \left( \sum_{j=1}^{N-1} \rho_j g h_j \right) \frac{\partial h_N}{\partial t} - \left( \sum_{j=1}^{N-2} \rho_j g h_j \right) \left( \sum_{j=N-1}^{N} \frac{\partial h_j}{\partial t} \right) \, . \tag{5.34}$$

For the layer-layer formulation, available potential energy transfer occurs between adjacent layers and transfer between kinetic energy and available potential energy occurs within a layer. In (5.34) the first term on the RHS is transfer between kinetic energy and available potential energy while the second term is the transfer of available potential energy between layers N and N-1. In the bottom layer there is no transfer between kinetic energy and available potential energy; both terms on the RHS of (5.33) correspond to a transfer of available potential energy between layers N and N-1. It is obvious that the terms for the transfer of available potential energy between layers N and N-1 in (5.33) and (5.34) do not match as they should. It is necessary to regroup the factors in (5.34) and, therefore, (5.27).

Define a new quantity, $\tilde{p}_i$, as

$$\tilde{p}_i = \hat{p}_i - \left( \frac{\rho_i}{\rho_N} \right) \hat{p}_N \, . \tag{5.35}$$

Note that $\tilde{p}_N = 0$. The substitution of (5.35) into (5.27) yields

$$\frac{\partial A_i}{\partial t} = \tilde{p}_i \frac{\partial h_i}{\partial t} + \left( \frac{\rho_i}{\rho_N} \right) \hat{p}_N \frac{\partial h_i}{\partial t} + \left( \sum_{j=1}^{i} \rho_j g h_j \right) \left( \sum_{j=i+1}^{N} \frac{\partial h_j}{\partial t} \right)$$

$$- \left( \sum_{j=1}^{i-1} \rho_j g h_j \right) \left( \sum_{j=i}^{N} \frac{\partial h_j}{\partial t} \right) \tag{5.36}$$

or

$$\frac{\partial A_i}{\partial t} = \tilde{p}_i \frac{\partial h_i}{\partial t} - \left(\frac{\hat{p}_N}{\rho_N}\right) \sum_{j=i+1}^{N} \rho_j \frac{\partial h_j}{\partial t} + \left( \sum_{j=1}^{i} \rho_j g h_j \right) \left( \sum_{j=i+1}^{N} \frac{\partial h_j}{\partial t} \right)$$

$$+ \left(\frac{\hat{p}_N}{\rho_N}\right) \sum_{j=i}^{N} \rho_j \frac{\partial h_j}{\partial t} - \left( \sum_{j=1}^{i-1} \rho_j g h_j \right) \left( \sum_{j=i}^{N} \frac{\partial h_j}{\partial t} \right) . \qquad (5.37)$$

For i=N, (5.37) becomes

$$\frac{\partial A_N}{\partial t} = \underset{(1)}{\hat{p}_N \frac{\partial h_N}{\partial t}} - \underset{(2)}{\left( \sum_{j=1}^{N-1} \rho_j g h_j \right) \frac{\partial h_N}{\partial t}} , \qquad (5.38)$$

which is the same as (5.33). For i=N-1, (5.37) becomes

$$\frac{\partial A_{N-1}}{\partial t} = \underset{(3)}{\tilde{p}_{N-1} \frac{\partial h_{N-1}}{\partial t}} - \underset{(1)}{\hat{p}_N \frac{\partial h_N}{\partial t}} + \underset{(2)}{\left( \sum_{j=1}^{N-1} \rho_j g h_j \right) \frac{\partial h_N}{\partial t}}$$

$$+ \underset{(4)}{\left(\frac{\hat{p}_N}{\rho_N}\right) \left( \sum_{j=N-1}^{N} \rho_j \frac{\partial h_j}{\partial t} \right)} - \underset{(5)}{\left( \sum_{j=1}^{N-2} \rho_j g h_j \right) \left( \sum_{j=N-1}^{N} \frac{\partial h_j}{\partial t} \right)} . \qquad (5.39)$$

Now, terms 1 and 2 in (5.39) match those on the RHS of (5.38), so these terms are the transfer of available potential energy between layers N and N-1. Terms 4 and 5 of (5.39) give the transfer of available potential energy between layers N-1 and N-2. Term 3 should correspond to the transfer between kinetic energy and available potential energy in layer N-1. Does such a matching term exist in the kinetic energy relation. Use (5.35) in (5.26) to replace $\hat{p}_i$:

$$\frac{\partial K_i}{\partial t} = - \vec{\nabla} \cdot \vec{v}_i \left( K_i + h_i \tilde{p}_i \right) - \tilde{p}_i \frac{\partial h_i}{\partial t} \qquad (5.40)$$

or, for layer N-1,

$$\frac{\partial K_{N-1}}{\partial t} = - \vec{\nabla}\cdot\vec{v}_{N-1} \left(K_{N-1} + h_{N-1}\tilde{p}_{N-1}\right) - \tilde{p}_{N-1} \frac{\partial h_{N-1}}{\partial t} \quad . \tag{5.41}$$

(6) (3)

Term 3 appears in (5.39) and (5.41) and is the transfer between kinetic and available potential energy within the layer N-1. For N-2, (5.37) and (5.40) become

$$\frac{\partial A_{N-2}}{\partial t} = \tilde{p}_{N-2} \frac{\partial h_{N-2}}{\partial t} - \left(\frac{\hat{p}_N}{\rho_N}\right) \left( \sum_{j=N-1}^{N} \rho_j \frac{\partial h_j}{\partial t} \right) + \left( \sum_{j=1}^{N-2} \rho_j g h_j \right) \left( \sum_{j=N-1}^{N} \frac{\partial h_j}{\partial t} \right)$$

(7) (4) (5)

$$+ \left(\frac{\hat{p}_N}{\rho_N}\right) \sum_{j=N-2}^{N} \rho_j \frac{\partial h_j}{\partial t} - \left( \sum_{j=1}^{N-3} \rho_j g h_j \right) \left( \sum_{j=N-2}^{N} \frac{\partial h_j}{\partial t} \right) \tag{5.42}$$

(8) (9)

and

$$\frac{\partial K_{N-2}}{\partial t} = - \vec{\nabla}\cdot\vec{v}_{N-2} \left(K_{N-2} + h_{N-2}\tilde{p}_{N-2}\right) - \tilde{p}_{N-2} \frac{\partial h_{N-2}}{\partial t} \quad . \tag{5.43}$$

(10) (7)

These relations are shown in the block diagram in Figure 5.

The preceding discussion indicates that kinetic energy and available potential energy relations for the reduced-gravity model can be written and interpreted as follows:

$$\frac{\partial K_i}{\partial t} = - \vec{\nabla}\cdot\vec{v}_i \left(K_i + h_i\tilde{p}_i\right) - \tilde{p}_i \frac{\partial h_i}{\partial t} \qquad i<N \tag{5.44}$$

(1) (2)

$$\frac{\partial K_N}{\partial t} = 0 \qquad i=N \tag{5.45}$$

$$\frac{\partial A_i}{\partial t} = \tilde{p}_i \frac{\partial h_i}{\partial t} - \left[ \left(\frac{\hat{p}_N}{\rho_N}\right) \sum_{j=i+1}^{N} \rho_j \frac{\partial h_j}{\partial t} - \left( \sum_{j=1}^{i} \rho_j g h_j \right) \left( \sum_{j=i+1}^{N} \frac{\partial h_j}{\partial t} \right) \right]$$

(2) (3)

$$+\left[\left(\frac{\hat{p}_N}{\rho_N}\right)\sum_{j=i}^{N}\rho_j\frac{\partial h_j}{\partial t}-\left(\sum_{j=1}^{i-1}\rho_j g h_j\right)\left(\sum_{j=i}^{N}\frac{\partial h_j}{\partial t}\right)\right] \qquad i\leq N \qquad (5.46)$$

(4)

Term 1 is the modified energy flux divergence. Term 2 is the transfer between kinetic energy and available potential energy within a layer. Term 3 is the transfer of available potential energy between layer i and the layer below. Term 4 is the transfer of available potential energy between layer i and the layer above.

Note that $\vec{v}_N$ and $\tilde{p}_N$ are zero, so for i=N, (5.44) becomes (5.36) or

$$\frac{\partial K_N}{\partial t} = 0 \ . \qquad (5.47)$$

However, it is desirable to apply (5.44) only for layers i<N because (5.6) has been used to get (5.44) and (5.6) is valid only for i<N.

The local time derivative of the total kinetic energy per unit area is found by summing (5.44) over N-1 layers and adding (5.45) to get

$$\sum_{i=1}^{N}\frac{\partial K_i}{\partial t} = -\sum_{i=1}^{N-1}\vec{\nabla}\cdot\vec{v}_i\left(K_i + h_i\tilde{p}_i\right) - \sum_{i=1}^{N-1}\tilde{p}_i\frac{\partial h_i}{\partial t} \qquad (5.48)$$

Summing (5.46) over N layers shows that the local time derivative of the total available potential energy per unit area is

$$\sum_{i=1}^{N}\frac{\partial A_i}{\partial t} = +\sum_{i=1}^{N}\tilde{p}_i\frac{\partial h_i}{\partial t} = \sum_{i=1}^{N-1}\tilde{p}_i\frac{\partial h_i}{\partial t} \ . \qquad (5.49)$$

Adding (5.48) and (5.49) gives the local time derivative of the total energy per unit area as

$$\sum_{i=1}^{N}\frac{\partial}{\partial t}\left(K_i + A_i\right) = -\sum_{i=1}^{N-1}\vec{\nabla}\cdot\vec{v}_i\left(K_i + h_i\tilde{p}_i\right) \ . \qquad (5.50)$$

A comparison of (5.50) and (2.32), that is the change w.r.t. time of the total energy per unit area of the system for finite depth model shows that the RHS of both equations appear to contain energy flux divergence terms. Note, however, that (5.48) is summed over only N-1 layers; the bottom layer does not contribute an energy flux divergence term in the reduced-gravity model. Furthermore, energy flux divergence terms for the reduced-gravity model use

$\tilde{p}_i$, not $\hat{p}_i$, as for the finite depth model. Use (5.35) to eliminate $\tilde{p}_i$ in (5.48). The result is

$$\sum_{i=1}^{N} \frac{\partial}{\partial t}\left(K_i + A_i\right) = -\sum_{i=1}^{N-1} \vec{\nabla}\cdot\vec{v}_i \left(K_i + h_i\hat{p}_i\right) + \frac{\hat{p}_N}{\rho_N}\sum_{i=1}^{N-1} \rho_i \vec{\nabla}\cdot\vec{v}_i h_i \ . \tag{5.51}$$

From (5.6) and (5.25), (5.51) can be rewritten as

$$\sum_{i=1}^{N} \frac{\partial}{\partial t}\left(K_i + A_i\right) = -\sum_{i=1}^{N-1} \vec{\nabla}\cdot\vec{v}_i \left(K_i + h_i\hat{p}_i\right) + \hat{p}_N \frac{\partial \eta_N}{\partial t} \ . \tag{5.52}$$

It is apparent from (5.52) that the energy flux divergence term for the bottom layer that is present in the finite depth model, (2.32), has been replaced by $\hat{p}_N\, \partial\eta_N/\partial t$ in the reduced-gravity model. Now integrate (5.52) over the horizontal area of the system to get the local time derivative of the total energy for the entire system of the reduced-gravity model:

$$\iint \sum_{i=1}^{N} \frac{\partial}{\partial t}\left(K_i + A_i\right)dxdy = -\iint \sum_{i=1}^{N-1} \vec{\nabla}\cdot\vec{v}_i \left(K_i + h_i\hat{p}_i\right)dxdy + \iint \hat{p}_N \frac{\partial \eta_N}{\partial t}dxdy \tag{5.53}$$

or

$$\frac{\partial}{\partial t}\iint \sum_{i=1}^{N} \left(K_i + A_i\right)dxdy = -\sum_{i=1}^{N} \int_{S_i} \left(K_i/h_i + \hat{p}_i\right) \vec{v}_i\cdot\hat{n}_i \, dS_i$$

$$+ \hat{p}_N \frac{\partial}{\partial t}\left\langle \eta_N \right\rangle \times \text{Area} \ , \tag{5.54}$$

where $S_i$ is the total surface of the side walls for layer i, $\hat{n}_i$ is the unit vector normal to the side walls and $dS_i$ represents the differential area of the side walls for layer i. Now from (5.16), $\langle \eta_N \rangle = 0$, so that last term on the RHS of (5.54) is zero. If the side walls are solid, there is no flux through them and the first term on the RHS of (5.54) vanishes, so

$$\frac{\partial}{\partial t}\iint \sum_{i=1}^{N} \left(K_i + A_i\right)dxdy = 0 \ . \tag{5.55}$$

This relation merely states that the total energy is conserved for an inviscid multilayer reduced-gravity hydrodynamic model with no external sources or sinks. Thus, as a comparison of (2.34) and (5.55) shows, the additional assumption of an infinitely deep bottom layer does not change the fact that total energy is conserved for a system.

The energy specifications and relations in the layer-layer formulation for the reduced-gravity model are given in sections C.11 and C.12 for the one and two active layer cases, respectively.

Part B, Section 6

In this section the specific equations for transfer of kinetic energy and available potential energy are found in a layer-interface formulation for an inviscid multilayer reduced-gravity hydrodynamic model. The reduced-gravity model is discussed extensively in section B.5 and the layer-interface formulation is treated in section B.3. Both sections should be perused before reading further in this section.

The starting equations, derived in the Appendix, remain valid for the first N-1 layers. Therefore, the same energy relations are valid for these layers. These relations, (3.26) - (3.28), are

$$\frac{\partial K_i}{\partial t} = - \vec{\nabla}\cdot\vec{v}_i \left(K_i + h_i \hat{p}_i\right) - \hat{p}_i \frac{\partial \eta_i}{\partial t} + \hat{p}_i \frac{\partial \eta_{i+1}}{\partial t} \qquad i<N \ , \tag{6.1}$$

$$\frac{\partial \tilde{A}_i}{\partial t} = \hat{p}_i \frac{\partial \eta_i}{\partial t} - \hat{p}_{i-1} \frac{\partial \eta_i}{\partial t} \qquad 1<i<N \ , \tag{6.2}$$

$$\frac{\partial \tilde{A}_1}{\partial t} = \hat{p}_1 \frac{\partial \eta_1}{\partial t} \qquad i=1 \ . \tag{6.3}$$

In a reduced-gravity model both the velocity, $\vec{v}_N$, and the kinetic energy, $K_N$, are zero in the bottom layer while the pressure, $p_N$, is constant. From (5.22), the pressure can be written as

$$\hat{p}_N = - \sum_{j=1}^{N-1} \left(\rho_N - \rho_j\right) g \, H_j \ . \tag{6.4}$$

Section B.5 shows that $\partial h_N / \partial t$ cannot be determined from (A.17), which is not valid in the bottom layer, but it can be found from (5.25) as

$$\frac{\partial h_N}{\partial t} = - \sum_{j=1}^{N-1} \left(\frac{\rho_j}{\rho_N}\right) \frac{\partial h_j}{\partial t} \tag{6.5}$$

or

$$\sum_{j=1}^{N} \rho_j \frac{\partial h_j}{\partial t} = 0 \ . \tag{6.6}$$

The available potential energy relation, (6.2), was derived in section B.3 and does not depend on the depth of the bottom layer. So, for i=N, (6.2) becomes

$$\frac{\partial \tilde{A}_N}{\partial t} = \underset{(1)}{\hat{p}_N \frac{\partial \eta_N}{\partial t}} - \underset{(2)}{\hat{p}_{N-1} \frac{\partial \eta_N}{\partial t}} . \tag{6.7}$$

Because the kinetic energy is constant and zero in the bottom layer, the kinetic energy relation for the bottom layer is

$$\frac{\partial K_N}{\partial t} = 0 . \tag{6.8}$$

A mathematically consistent set of energy relations has been derived for a reduced-gravity model in the layer-interface formulation: (6.1) - (6.3) for layers 1 to N-1 and (6.7) and (6.8) for layer N. However, as is the case for the layer-layer formulation, there is a problem in interpreting the energy transfer terms. To illustrate this problem, note that for i=N-1, (6.1) and (6.2) become

$$\frac{\partial K_{N-1}}{\partial t} = \underset{(3)}{- \vec{\nabla}\cdot\vec{v}_{N-1}\left(K_{N-1} + h_{N-1}\hat{p}_{N-1}\right)} \underset{(4)}{- \hat{p}_{N-1} \frac{\partial \eta_{N-1}}{\partial t}} \underset{(2)}{+ \hat{p}_{N-1} \frac{\partial \eta_N}{\partial t}} \tag{6.9}$$

and

$$\frac{\partial \tilde{A}_{N-1}}{\partial t} = \underset{(4)}{\hat{p}_{N-1} \frac{\partial \eta_{N-1}}{\partial t}} - \underset{(5)}{\hat{p}_{N-2} \frac{\partial \eta_{N-1}}{\partial t}} . \tag{6.10}$$

Term 2 appears to be the transfer between available potential energy in interface N and the kinetic energy in layer N-1. Term 4 appears to be the transfer between the kinetic energy in layer N-1 and the available potential energy in interface N-1. What is term 1 in (6.7)? It must be a term to transfer energy upward, but there is no matching term in (6.9) and (6.10). It is obvious that the terms need to be regrouped in (6.1), (6.2) and (6.7).

In section B.5, a new quantity, $\tilde{p}_i$, is defined in (5.35) as

$$\tilde{p}_i = \hat{p}_i - \left(\frac{\rho_i}{\rho_N}\right) \hat{p}_N , \tag{6.11}$$

where $\hat{p}_N$ is a constant in the reduced-gravity model and $\tilde{p}_N = 0$. Substituting (6.11) into (6.1) and (6.2) gives

$$\frac{\partial K_i}{\partial t} = - \vec{\nabla}\cdot\vec{v}_i \left(K_i + h_i \tilde{p}_i\right) - \tilde{p}_i \frac{\partial \eta_i}{\partial t} + \tilde{p}_i \frac{\partial \eta_{i+1}}{\partial t} , \tag{6.12}$$

$$\frac{\partial \tilde{A}_i}{\partial t} = + \tilde{p}_i \frac{\partial \eta_i}{\partial t} - \tilde{p}_{i-1} \frac{\partial \eta_i}{\partial t} + \left(\frac{\rho_i}{\rho_N}\right) \hat{p}_N \frac{\partial \eta_i}{\partial t} - \left(\frac{\rho_{i-1}}{\rho_N}\right) \hat{p}_N \frac{\partial \eta_i}{\partial t} \tag{6.13}$$

or

$$\frac{\partial \tilde{A}_i}{\partial t} = \tilde{p}_i \frac{\partial \eta_i}{\partial t} - \tilde{p}_{i-1} \frac{\partial \eta_i}{\partial t} + \left(\frac{\hat{p}_N}{\rho_N}\right) \sum_{j=i}^{N} \left(\rho_j - \rho_{j-1}\right) \frac{\partial \eta_j}{\partial t}$$

$$- \left(\frac{\hat{p}_N}{\rho_N}\right) \sum_{j=i+1}^{N} \left(\rho_j - \rho_{j-1}\right) \frac{\partial \eta_j}{\partial t} . \tag{6.14}$$

For i=N-2, (6.14) and (6.12) become

$$\frac{\partial \tilde{A}_{N-2}}{\partial t} = \underset{(1)}{\tilde{p}_{N-2} \frac{\partial \eta_{N-2}}{\partial t}} - \underset{(2)}{\tilde{p}_{N-3} \frac{\partial \eta_{N-2}}{\partial t}} + \underset{(3)}{\left(\frac{\hat{p}_N}{\rho_N}\right) \sum_{j=N-2}^{N} \left(\rho_j - \rho_{j-1}\right) \frac{\partial \eta_j}{\partial t}}$$

$$\underset{(4)}{- \left(\frac{\hat{p}_N}{\rho_N}\right) \sum_{j=N-1}^{N} \left(\rho_j - \rho_{j-1}\right) \frac{\partial \eta_j}{\partial t}} \tag{6.15}$$

and

$$\frac{\partial K_{N-2}}{\partial t} = \underset{(5)}{- \vec{\nabla}\cdot\vec{v}_{N-2} \left(K_{N-2} + h_{N-2}\tilde{p}_{N-2}\right)} \underset{(1)}{- \tilde{p}_{N-2} \frac{\partial \eta_{N-2}}{\partial t}} + \underset{(6)}{\tilde{p}_{N-2} \frac{\partial \eta_{N-1}}{\partial t}} . \tag{6.16}$$

For i=N-1, (6.14) and (6.12) become

$$\frac{\partial \tilde{A}_{N-1}}{\partial t} = \underset{(7)}{\tilde{p}_{N-1}} \frac{\partial \eta_{N-1}}{\partial t} - \underset{(6)}{\tilde{p}_{N-2}} \frac{\partial \eta_{N-1}}{\partial t} + \left(\frac{\hat{p}_N}{\rho_N}\right) \underset{j=N-1}{\overset{N}{\Sigma}} \underset{(4)}{\left(\rho_j - \rho_{j-1}\right)} \frac{\partial \eta_j}{\partial t}$$

$$- \left(\frac{\hat{p}_N}{\rho_N}\right) \underset{(8)}{\left(\rho_N - \rho_{N-1}\right)} \frac{\partial \eta_N}{\partial t} \qquad (6.17)$$

and

$$\frac{\partial K_{N-1}}{\partial t} = - \underset{(9)}{\vec{\nabla}\cdot\vec{v}_{N-1}} \underset{(7)}{\left(K_{N-1} + h_{N-1}\tilde{p}_{N-1}\right)} - \underset{(7)}{\tilde{p}_{N-1}} \frac{\partial \eta_{N-1}}{\partial t} + \underset{(10)}{\tilde{p}_{N-1}} \frac{\partial \eta_N}{\partial t} . \qquad (6.18)$$

Substituting (6.11) into (6.7) and (6.8) yields

$$\frac{\partial \tilde{A}_N}{\partial t} = - \underset{(10)}{\tilde{p}_{N-1}} \frac{\partial \eta_N}{\partial t} + \left(\frac{\hat{p}_N}{\rho_N}\right) \underset{(8)}{\left(\rho_N - \rho_{N-1}\right)} \frac{\partial \eta_N}{\partial t} \qquad (6.19)$$

$$\frac{\partial K_N}{\partial t} = 0 . \qquad (6.20)$$

Terms 5 and 9 are the modified energy flux divergences for layers N-2 and N-1, respectively. Terms 1 and 7 are the transfer between kinetic energy in a given layer and available potential energy of the interface directly above for layers N-2 and N-1, respectively. Terms 6 and 10 give the transfer between kinetic energy in a given layer and available potential energy of the interface directly below for layers N-2 and N-1, respectively. Terms 8 and 4 represent the transfer of available potential energy between a given interface and the interface above for interfaces N and N-1, respectively. These relations are shown in the block diagram in Figure 6.

A comparison of Figure 6 to Figure 3 shows that the introduction of the reduced-gravity assumption has changed the types of energy transfer possible in the layer-interface formulation. In Figure 3, there is no transfer of available potential energy between interfaces. It seems physically reasonable for energy to be transferred between a layer and an adjacent interface. The interface deviations are not directly related for the model in section B.3.

In Figure 6, there is a transfer of available potential energy between consecutive interfaces. This coupling occurs in the reduced-gravity model because the interface deviations are directly related through (6.6) which is valid only for the reduced-gravity model.

The preceding discussion indicates that kinetic energy and available potential energy relations for the reduced-gravity model in layer-interface formulation can be written and interpreted as follows:

$$\frac{\partial K_i}{\partial t} = - \vec{\nabla}\cdot\vec{v}_i \left(K_i + h_i \tilde{p}_i\right) - \tilde{p}_i \frac{\partial \eta_i}{\partial t} + \tilde{p}_i \frac{\partial \eta_{i+1}}{\partial t} \qquad i<N \ , \tag{6.21}$$

(1) (2) (3)

$$\frac{\partial K_N}{\partial t} = 0 \qquad i=N \ , \tag{6.22}$$

$$\frac{\partial \tilde{A}_i}{\partial t} = \tilde{p}_i \frac{\partial \eta_i}{\partial t} - \tilde{p}_{i-1} \frac{\partial \eta_i}{\partial t} + \left(\frac{\hat{p}_N}{\rho_N}\right) \sum_{j=i}^{N} \left(\rho_j - \rho_{j-1}\right) \frac{\partial \eta_j}{\partial t}$$

(2) (4) (5)

$$- \left(\frac{\hat{p}_N}{\rho_N}\right) \sum_{j=i+1}^{N} \left(\rho_j - \rho_{j-1}\right) \frac{\partial \eta_j}{\partial t} \qquad 1 \le i \le N \ . \tag{6.23}$$

(6)

Term 2 is the transfer between the kinetic energy in layer i and the available potential energy for the interface i, directly above. Term 3 and term 4 are the same type of transfer for layer i and interface i+1 and for layer i-1 and interface i, respectively. Terms 5 and 6 represent a transfer of available potential energy from interface i to the interface above and below, respectively. Finally, term 1 is the modified energy flux divergence.

Summing (6.21) over N-1 layers and adding (6.22) gives the time derivative of the total kinetic energy per unit area for the reduced-gravity model as

$$\sum_{i=1}^{N} \frac{\partial K_i}{\partial t} = - \sum_{i=1}^{N-1} \vec{\nabla}\cdot\vec{v}_i \left(K_i + h_i \tilde{p}_i\right) - \sum_{i=1}^{N-1} \tilde{p}_i \frac{\partial}{\partial t} \left(\eta_i - \eta_{i+1}\right) \ . \tag{6.24}$$

Summing (6.23) over N interfaces gives the time derivative of the total interface available potential energy per unit area for the reduced-gravity model as

$$\sum_{i=1}^{N} \frac{\partial \tilde{A}_i}{\partial t} = \sum_{i=1}^{N} \tilde{p}_i \frac{\partial \eta_i}{\partial t} - \sum_{i=1}^{N} \tilde{p}_{i-1} \frac{\partial \eta_i}{\partial t} \tag{6.25}$$

or, rearranging terms,

$$\sum_{i=1}^{N} \frac{\partial \tilde{A}_i}{\partial t} = + \sum_{i=1}^{N-1} \tilde{p}_i \frac{\partial}{\partial t} \left( \eta_i - \eta_{i+1} \right) . \tag{6.26}$$

Adding (6.24) and (6.26) yields the time derivative of the total energy per unit area for the reduced-gravity model as

$$\sum_{i=1}^{N} \frac{\partial}{\partial t} \left( K_i + \tilde{A}_i \right) = - \sum_{i=1}^{N-1} \vec{\nabla} \cdot \vec{v}_i \left( K_i + h_i \tilde{p}_i \right) \tag{6.27}$$

in the layer-interface formulation. Eliminate $\tilde{p}_i$ in (6.27) by using (6.11), (5.6), and (5.25); the result is

$$\sum_{i=1}^{N} \frac{\partial}{\partial t} \left( K_i + \tilde{A}_i \right) = - \sum_{i=1}^{N-1} \vec{\nabla} \cdot \vec{v}_i \left( K_i + h_i \hat{p}_i \right) + \hat{p}_N \frac{\partial \eta_N}{\partial t} . \tag{6.28}$$

As discussed in section B.5 following (5.52) the energy flux divergence term for the bottom layer in the finite depth model, (2.32), is replaced by the last term on the RHS of (6.28). Now integrate (6.28) over the horizontal area of the basin to get the local time derivative of the total energy in the basin for the reduced-gravity model:

$$\frac{\partial}{\partial t} \iint \sum_{i=1}^{N} \left( K_i + \tilde{A}_i \right) dxdy = - \iint \sum_{i=i}^{N-1} \vec{\nabla} \cdot \vec{v}_i \left( K_i + h_i \hat{p}_i \right) dxdy + \hat{p}_N \frac{\partial \langle \eta_N \rangle}{\partial t} \times \text{Area} \tag{6.29}$$

or

$$\frac{\partial}{\partial t} \iint \sum_{i=1}^{N} \left( K_i + \tilde{A}_i \right) dxdy = - \int_{S_i} \left( K_i / h_i + \hat{p}_i \right) \vec{v}_i \cdot \hat{n}_i \, dS_i . \tag{6.30}$$

If the side walls are solid, then there is no flux through the side walls and (6.30) becomes

$$\frac{\partial}{\partial t} \iint \sum_{i=1}^{N} \left(K_i + \tilde{A}_i\right) dxdy = 0 \ . \qquad (6.31)$$

This relation indicates that in the layer-interface formulation the total energy in the basin is conserved for the reduced-gravity model as expected from (5.55).

The energy specifications and relations in the layer-interface formulation for the reduced-gravity model are given in sections D.16 and D.17 for the one and two active layer cases, respectively.

The energy relations developed in sections B.1 - B.3 for the finite depth model and in sections B.5 - B.6 for the reduced-gravity model illustrate two problems common to all energy analysis in geophysical fluid dynamics: first, specification of the energies present and, second, grouping of energy transfer terms so that realistic physical mechanisms are presented. Unfortunately, no foolproof method exists for handling these two problems. In fact, the work in section B.4 is included to show how easy it is to develop a mathematically correct,, but physically unsatisfactory, set of energy relations.

Specification of the energies may seem straightforward at first, particularly for hydrodynamic models that use only kinetic and potential energies. However, only that portion of the potential energy that can be converted into kinetic energy is of interest in an energy analysis. Thus, in section B.1 this available potential energy is specified. In general, the specification of available potential energy and available internal energy, if present, can be difficult, particularly for those models that include thermodynamic effects or stratification. For further information see Lorenz (1967) and Pedlosky (1987) among others.

Specification of the energies also depends on the volume of fluid being analyzed. For example, is the energy analysis to be made for a unit volume of fluid, a column of fluid, or for the entire region being studied? For multilayer hydrodynamic models it seems logical to integrate kinetic and available potential energy densities over the layer depth to obtain the kinetic and available potential energies per unit area for each layer. This procedure is carried out in sections B.1 and B.2 for the finite depth model and is called the layer-layer formulation. However, as shown in section B.3, it is also possible to associate the available energy with a given interface; this procedure is called the layer-interface formulation. Similarly, for the reduced-gravity model, available potential energy per unit area for a layer or for an interface may be used as shown in sections B.5 and B.6.

The second problem in energy analysis, the grouping of energy transfer terms, is illustrated for the finite depth model by comparing the kinetic energy relations for the layer-layer formulation, (2.27), and for the layer-interface formulation, (3.26). Both formulations use the kinetic energy per unit area for a layer, (1.1), and start with the same equation for the local time derivative of kinetic energy per unit, (1.11). Thus, both the kinetic energy per unit area in a layer and the change w.r.t. time of this quantity have the same magnitude in the two formulations. However, the layer-layer formulation has only one exchange term between kinetic and available potential energy but the layer-interface formulation has two.

The role of energetics in understanding the physical mechanisms present in various models is demonstrated clearly in the comparison of the finite depth and reduced-gravity models. Consider the layer-layer formulation of the energetics. A comparison of the energy block diagrams (Figs. 2 and 5) shows that the same type of energy transfers occur in both models but a comparison of the energy relations, (2.27) - (2.28) and (5.44) - (5.46), shows that the magnitudes of these transfer terms are different. The situation is clarified

further when the layer-interface formulation is examined. A comparison of the energy block diagrams (Figs. 3 and 6) shows that the energy transfers in this formulation are different for the two models. Specifically, in the reduced-gravity model interface available potential energy is transferred between consecutive interfaces, but no such transfer occurs in the finite depth model. This direct transfer of interface available potential energy between consecutive interfaces is a consequence of the fact that the pressure in the bottom layer of a reduced-gravity model must be independent of horizontal position and time. This independence, in turn, dictates that the interface deviation at the top of the bottom layer, $\eta_N$, is determined completely by the density structure and the rest of the interface deviations.

Part C, Section 8

The energy relations in the layer-layer formulation are given in this section for the special case of the one-layer, finite depth model. A vertical cross-section of the one-layer model is shown in Figure 7.

For the one-layer model, N=1 and i=1. From (1.1) and (1.22), the kinetic and available potential energies are

$$K_1 = \frac{1}{2} h_1 \rho_1 \left( u_1^2 + v_1^2 \right) \tag{8.1}$$

and

$$A_1 = \frac{1}{2} \rho_1 g \eta_1^2 \ . \tag{8.2}$$

Because there is only one layer, the total kinetic energy per unit area and the total available potential energy per unit area are represented by (8.1) and (8.2), respectively.

From (2.27) and (2.28), for i=1 and N=1, the energy relations for a one layer, finite depth model are

$$\frac{\partial K_1}{\partial t} = \underbrace{- \vec{\nabla} \cdot \vec{v}_1 \left( K_1 + h_1 \hat{p}_1 \right)}_{(1)} \underbrace{- \hat{p}_1 \frac{\partial h_1}{\partial t}}_{(2)} \tag{8.3}$$

and

$$\frac{\partial A_1}{\partial t} = \underbrace{\hat{p}_1 \frac{\partial h_1}{\partial t}}_{(2)} \ , \tag{8.4}$$

where, from (2.5) and Figure 7,

$$\hat{p}_1 = \rho_1 g \eta_1 = \rho_1 g \left( h_1 - H_1 \ . \right) \tag{8.5}$$

Term 1 is the energy flux divergence and term 2 is the exchange between kinetic energy and available potential energy within the layer. These relations are shown in the block diagram in Figure 8.

Adding (8.3) and (8.4) gives the local time derivative of the total energy per unit area in layer 1 as

$$\frac{\partial}{\partial t}\left(K_1 + A_1\right) = -\vec{\nabla}\cdot\vec{v}_1\left(K_1 + h_1\hat{p}_1\right) . \tag{8.6}$$

For the one layer model, (8.6) also represents the local time derivative of the total energy per unit area of the system. As expected from (2.32), the only term on the RHS is the energy flux divergence.

Part C, Section 9

In this section the energy relations are given for the special case of a two-layer, finite depth model in the layer-layer formulation. A vertical cross-section of the two-layer model is shown in Figure 9.

From (1.1) and (1.22), the kinetic and available potential energies per unit area in a layer for N=2 and i=1,2 are

$$K_1 = \frac{1}{2} h_1 \rho_1 \left(u_1^2 + v_1^2\right) , \tag{9.1}$$

$$A_1 = \frac{1}{2} \rho_1 g \left[\eta_1^2 - \left(\eta_2^2 - 2\eta_2 H_1\right)\right] , \tag{9.2}$$

$$K_2 = \frac{1}{2} h_2 \rho_2 \left(u_2^2 + v_2^2\right) , \tag{9.3}$$

$$A_2 = \frac{1}{2} \rho_2 g \left(\eta_2^2 - 2\eta_2 H_1\right) . \tag{9.4}$$

The total kinetic energy per unit area for the two-layer, finite depth model is found by adding (9.1) and (9.3) to get

$$K_1 + K_2 = \frac{1}{2} h_1 \rho_1 \left(u_1^2 + v_1^2\right) + \frac{1}{2} h_2 \rho_2 \left(u_2^2 + v_2^2\right) . \tag{9.5}$$

Similarly the total available potential energy per unit area for the two-layer, finite depth model is found by adding (9.2) and (9.4) to find

$$A_1 + A_2 = \frac{1}{2} \rho_1 g \eta_1^2 + \frac{1}{2} \left(\rho_2 - \rho_1\right) g \left(\eta_2^2 - 2\eta_2 H_1\right) . \tag{9.6}$$

From (2.27) and (2.28), for i=1,2 and N=2, the energy relations for a two-layer, finite depth model are

$$\frac{\partial K_1}{\partial t} = - \vec{\nabla}\cdot\vec{v}_1 \underbrace{\left(K_1 + h_1 \hat{p}_1\right)}_{(1)} - \underbrace{\hat{p}_1 \frac{\partial h_1}{\partial t}}_{(2)} , \tag{9.7}$$

$$\frac{\partial A_1}{\partial t} = \underbrace{\hat{p}_1 \frac{\partial h_1}{\partial t}}_{(2)} + \underbrace{\rho_1 g h_1 \frac{\partial h_2}{\partial t}}_{(3)} , \tag{9.8}$$

$$\frac{\partial K_2}{\partial t} = - \vec{\nabla}\cdot\vec{v}_2 \underbrace{\left(K_2 + h_2\hat{p}_2\right)}_{(4)} - \hat{p}_2 \underbrace{\frac{\partial h_2}{\partial t}}_{(5)} , \tag{9.9}$$

$$\frac{\partial A_2}{\partial t} = \hat{p}_2 \underbrace{\frac{\partial h_2}{\partial t}}_{(5)} - \rho_1 g \underbrace{h_1 \frac{\partial h_2}{\partial t}}_{(3)} , \tag{9.10}$$

where the pressures, $\hat{p}_1$ and $\hat{p}_2$, are given from (2.5) for i=1,2 as

$$\hat{p}_1 = \rho_1 g \eta_1 \tag{9.11}$$

$$\hat{p}_2 = \rho_2 g \eta_1 - \left(\rho_2 - \rho_1\right) g\, h_1 . \tag{9.12}$$

The energy transfers in (9.7) - (9.10) are shown in the block diagram in Figure 10.

Summing (9.7) and (9.9) gives the local time derivative of total kinetic energy per unit area for the system for the two-layer, finite depth model as

$$\frac{\partial}{\partial t}\left(K_1 + K_2\right) = \underbrace{- \vec{\nabla}\cdot\vec{v}_1 \left(K_1 + h_1\hat{p}_1\right)}_{(1)} \underbrace{- \hat{p}_1 \frac{\partial h_1}{\partial t}}_{(2)} \underbrace{- \vec{\nabla}\cdot\vec{v}_2 \left(K_2 + h_2\hat{p}_2\right)}_{(4)} \underbrace{- \hat{p}_2 \frac{\partial h_2}{\partial t}}_{(5)} . \tag{9.13}$$

Summing (9.8) and (9.10) gives the change in total available potential energy per area for the two-layer model as

$$\frac{\partial}{\partial t}\left(A_1 + A_2\right) = \underbrace{\hat{p}_1 \frac{\partial h_1}{\partial t}}_{(2)} + \underbrace{\hat{p}_2 \frac{\partial h_2}{\partial t}}_{(5)} . \tag{9.14}$$

Note that (9.14) has no term 3, which is the transfer of available potential energy between the two layers. Alternatively, the change in total energy per unit area in a layer for layers 1 and 2 can be found by adding (9.7) to (9.8) and (9.9) to (9.10), respectively, to get

$$\frac{\partial}{\partial t}\left(K_1 + A_1\right) = \underbrace{- \vec{\nabla}\cdot\vec{v}_1 \left(K_1 + h_1\hat{p}_1\right)}_{(1)} + \underbrace{\rho_1 g h_1 \frac{\partial h_2}{\partial t}}_{(3)} \tag{9.15}$$

and

$$\frac{\partial}{\partial t}\left(K_2 + A_2\right) = -\vec{\nabla}\cdot\vec{v}_2\left(K_2 + h_2\hat{p}_2\right) - \rho_1 g h_1 \frac{\partial h_2}{\partial t} \tag{9.16}$$

(4) (3)

Here the exchanges between kinetic and available potential energy with a layer, terms (2) and (5), are missing. The change with time in total energy per unit area for the two-layer, finite depth model is found by summing (9.7) - (9.10) to get

$$\frac{\partial}{\partial t}\left(K_1 + K_2 + A_1 + A_2\right) = -\vec{\nabla}\cdot\vec{v}_1\left(K_1 + h_1\hat{p}_1\right) - \vec{\nabla}\cdot\vec{v}_2\left(K_2 + h_2\hat{p}_2\right) . \tag{9.17}$$

(1) (4)

As expected from (2.32), the only terms on the RHS of (9.17) are the energy flux divergences for the two layers.

Part C, Section 10

In this section the energy relations are given for the special case of a three-layer model in the layer-layer formulation. A vertical cross-section of the three-layer model is shown in Figure 11.

From (1.1) and (1.22), the kinetic and available potential energies per area in a layer for N=3 and i=1,2,3, are

$$K_1 = \frac{1}{2} h_1 \rho_1 \left(u_1^2 + v_1^2\right) , \tag{10.1}$$

$$A_1 = \frac{1}{2} \rho_1 g \left[\eta_1^2 - \left(\eta_2^2 - 2\eta_2 H_1\right)\right] , \tag{10.2}$$

$$K_2 = \frac{1}{2} h_2 \rho_2 \left(u_2^2 + v_2^2\right) , \tag{10.3}$$

$$A_2 = \frac{1}{2} \rho_2 g \left[\left(\eta_2^2 - 2\eta_2 H_1\right) - \left(\eta_3^2 - 2\eta_3 \left(H_1 + H_2\right)\right)\right] , \tag{10.4}$$

$$K_3 = \frac{1}{2} h_3 \rho_3 \left(u_3^2 + v_3^2\right) , \tag{10.5}$$

$$A_3 = \frac{1}{2} \rho_3 g \left[\left(\eta_3^2 - 2\eta_3 \left(H_1 + H_2\right)\right)\right] . \tag{10.6}$$

The total kinetic energy per unit area for the three-layer, finite depth model is found by adding (10.1), (10.3) and (10.5) to get

$$K_1 + K_2 + K_3 = \frac{1}{2} h_1 \rho_1 \left(u_1^2 + v_1^2\right) + \frac{1}{2} h_2 \rho_2 \left(u_2^2 + v_2^2\right) + \frac{1}{2} h_3 \rho_3 \left(u_3^2 + v_3^2\right) \tag{10.7}$$

Similarly, the total available potential energy per unit area for the three-layer, finite depth model is found by adding (10.2), (10.4) and (10.6) to get

$$A_1 + A_2 + A_3 = \frac{1}{2} \rho_1 g \eta_1^2 + \frac{1}{2} \left(\rho_2 - \rho_1\right) g \left(\eta_2^2 - 2\eta_2 H_1\right) + \frac{1}{2} \left(\rho_3 - \rho_2\right) g \left[\eta_3^2 - 2\eta_3 \left(H_1 + H_2\right)\right] \tag{10.8}$$

From (2.27) and (2.28), for i=1,2,3 and N=3, the energy relations for a three-layer, finite depth model are

$$\frac{\partial K_1}{\partial t} = - \vec{\nabla}\cdot\vec{v}_1 \underset{(1)}{\left(K_1 + h_1\hat{p}_1\right)} - \underset{(2)}{\hat{p}_1 \frac{\partial h_1}{\partial t}} , \tag{10.9}$$

$$\frac{\partial A_1}{\partial t} = \underset{(2)}{\hat{p}_1 \frac{\partial h_1}{\partial t}} + \underset{(3)}{\left(\rho_1 g h_1\right)\left(\frac{\partial h_2}{\partial t} + \frac{\partial h_3}{\partial t}\right)} , \tag{10.10}$$

$$\frac{\partial K_2}{\partial t} = - \vec{\nabla}\cdot\vec{v}_2 \underset{(4)}{\left(K_2 + h_2\hat{p}_2\right)} - \underset{(5)}{\hat{p}_2 \frac{\partial h_2}{\partial t}} , \tag{10.11}$$

$$\frac{\partial A_2}{\partial t} = \underset{(5)}{\hat{p}_2 \frac{\partial h_2}{\partial t}} + \underset{(6)}{\left(\rho_1 g h_1 + \rho_2 g h_2\right)\frac{\partial h_3}{\partial t}} - \underset{(3)}{\left(\rho_1 g h_1\right)\left(\frac{\partial h_2}{\partial t} + \frac{\partial h_3}{\partial t}\right)} , \tag{10.12}$$

$$\frac{\partial K_3}{\partial t} = - \vec{\nabla}\cdot\vec{v}_3 \underset{(7)}{\left(K_3 + h_3\hat{p}_3\right)} - \underset{(8)}{\hat{p}_3 \frac{\partial h_3}{\partial t}} , \tag{10.13}$$

$$\frac{\partial A_3}{\partial t} = \underset{(8)}{\hat{p}_3 \frac{\partial h_3}{\partial t}} - \underset{(6)}{\left(\rho_1 g h_1 + \rho_2 g h_2\right)\frac{\partial h_3}{\partial t}} , \tag{10.14}$$

where the pressures, $\hat{p}_1$, $\hat{p}_2$, $\hat{p}_3$, are given from (2.5) for i=1,2,3 as

$$\hat{p}_1 = \rho_1 g \eta_1 , \tag{10.15}$$

$$\hat{p}_2 = \rho_2 g \eta_1 - \left(\rho_2 - \rho_1\right) g h_1 , \tag{10.16}$$

$$\hat{p}_3 = \rho_3 g \eta_1 - \left(\rho_3 - \rho_1\right) g h_1 - \left(\rho_3 - \rho_2\right) g h_2 . \tag{10.17}$$

These energy relations, (10.9) - (10.14), are shown in the block diagram in Figure 12.

Summing (10.9), (10.11) and (10.13) gives the change in the total kinetic energy per unit area of the system for the three-layer, finite depth model as

$$\frac{\partial}{\partial t}\left(K_1 + K_2 + K_3\right) = -\vec{\nabla}\cdot\vec{v}_1\left(K_1 + h_1\hat{p}_1\right) - \hat{p}_1\frac{\partial h_1}{\partial t} - \vec{\nabla}\cdot\vec{v}_2\left(K_2 + h_2\hat{p}_2\right)$$

(1) (2) (4)

$$- \hat{p}_2\frac{\partial h_2}{\partial t} - \vec{\nabla}\cdot\vec{v}_3\left(K_3 + h_3\hat{p}_3\right) - \hat{p}_3\frac{\partial h_3}{\partial t} \qquad (10.18)$$

(5) (7) (8)

Similarly, summing (10.10), (10.12) and (10.14) gives the change in the total available potential energy per unit area for the three-layer model as

$$\frac{\partial}{\partial t}\left(A_1 + A_2 + A_3\right) = \hat{p}_1\frac{\partial h_1}{\partial t} + \hat{p}_2\frac{\partial h_2}{\partial t} + \hat{p}_3\frac{\partial h_3}{\partial t} \qquad (10.19)$$

(2) (5) (8)

Alternatively, the change in total energy per unit area in a layer for layers 1-3 can be found by adding (10.9) to (10.10), (10.11) to (10.12), and (10.13) to (10.14), respectively, to get

$$\frac{\partial}{\partial t}\left(K_1 + A_1\right) = -\vec{\nabla}\cdot\vec{v}_1\left(K_1 + h_1\hat{p}_1\right) + \left(\rho_1 g h_1\right)\left(\frac{\partial h_2}{\partial t} + \frac{\partial h_3}{\partial t}\right) \qquad (10.20)$$

(1) (3)

$$\frac{\partial}{\partial t}\left(K_2 + A_2\right) = -\vec{\nabla}\cdot\vec{v}_2\left(K_2 + h_2\hat{p}_2\right) + \left(\rho_1 g h_1 + \rho_2 g h_2\right)\frac{\partial h_3}{\partial t}$$

(4) (6)

$$- \left(\rho_1 g h_1\right)\left(\frac{\partial h_2}{\partial t} + \frac{\partial h_3}{\partial t}\right) \qquad (10.21)$$

(3)

$$\frac{\partial}{\partial t}\left(K_3 + A_3\right) = -\vec{\nabla}\cdot\vec{v}_3\left(K_3 + h_3\hat{p}_3\right) - \left(\rho_1 g h_1 + \rho_2 g h_2\right)\frac{\partial h_3}{\partial t} \qquad (10.22)$$

(7) (6)

The change in the total energy per unit area for the three-layer model is found by summing (10.9) - (10.14) to get

$$\frac{\partial}{\partial t}\left(K_1 + A_1 + K_2 + A_2 + K_3 + A_3\right) = -\vec{\nabla}\cdot\vec{v}_1\left(K_1 + h_1\hat{p}_1\right)$$

(1)

$$-\vec{\nabla}\cdot\vec{v}_2\left(K_2 + h_2\,\hat{p}_2\right) - \vec{\nabla}\cdot\vec{v}_3\left(K_3 + h_3\,\hat{p}_3\right). \qquad (10.23)$$

(4) (7)

Again, as expected from (2.32), the only terms on the RHS of (10.23) are the energy flux divergences for the three layers.

Part C, Section 11

In this section the energy relations are given for the special case of a one active layer, reduced-gravity model in the layer-layer formulations. A vertical cross-section of the one active layer, reduced-gravity model is shown in Figure 13.

From (1.22) for i=1,2 and N=2

$$A_1 = \frac{1}{2}\rho_1 g\left[\eta_1^2 - \left(\eta_2^2 - 2\eta_2 H_1\right)\right] \tag{11.1}$$

and

$$A_2 = \frac{1}{2}\rho_2 g\left(\eta_2^2 - 2\eta_2 H_1\right)\Big] . \tag{11.2}$$

From (1.1) for i=1, and from the discussion following (5.3), the kinetic energy per unit area in layers 1 and 2 can be written as

$$K_1 = \frac{1}{2} h_1 \rho_1 \left(u_1^2 + v_1^2\right) \tag{11.3}$$

and

$$K_2 = 0 . \tag{11.4}$$

The total kinetic energy per unit area for the one active layer, reduced-gravity model is found by adding (11.3) and (11.4) to get

$$K_1 + K_2 = \frac{1}{2}\rho_1 h_1 \left(u_1^2 + v_1^2\right) . \tag{11.5}$$

Similarly the total available potential energy per unit area for the one active layer, reduced-gravity model is found by adding (11.1) and (11.2) to get

$$A_1 + A_2 = \frac{1}{2}\rho_1 g\,\eta_1^2 + \frac{1}{2}\left(\rho_2 - \rho_1\right)g\left(\eta_2^2 - 2\eta_2 H_1\right) . \tag{11.6}$$

From (5.44) - (5.46), for i=1,2 and N=2, the energy relations for a one active layer, reduced-gravity model are

$$\frac{\partial K_1}{\partial t} = - \vec{\nabla}\cdot\vec{v}_1 \underset{(1)}{\left(K_1 + h_1\tilde{p}_1\right)} - \underset{(2)}{\tilde{p}_1 \frac{\partial h_1}{\partial t}} , \tag{11.7}$$

$$\frac{\partial A_1}{\partial t} = \underset{(2)}{\tilde{p}_1 \frac{\partial h_1}{\partial t}} - \underset{(3)}{\left[\hat{p}_2 \frac{\partial h_2}{\partial t} - \rho_1 g h_1 \frac{\partial h_2}{\partial t}\right]} , \tag{11.8}$$

$$\frac{\partial K_2}{\partial t} = 0 , \tag{11.9}$$

$$\frac{\partial A_2}{\partial t} = \underset{(3)}{\left[\hat{p}_2 \frac{\partial h_2}{\partial t} - \rho_1 g h_1 \frac{\partial h_2}{\partial t}\right]} , \tag{11.10}$$

where, from (5.35) and (2.5), for i=1

$$\tilde{p}_1 = \rho_1 g \eta_1 - \left(\frac{\rho_1}{\rho_2}\right) \hat{p}_2 \tag{11.11}$$

and, from (5.22) for N=2

$$\hat{p}_2 = \left(\rho_1 - \rho_2\right) g H_1 . \tag{11.12}$$

The energy transfers in (11.7) - (11.10) are shown in the block diagram in Figure 14.

Adding (11.7) and (11.9) gives the change in the total kinetic energy per unit area for the one active layer, reduced-gravity model as

$$\frac{\partial}{\partial t}\left(K_1 + K_2\right) = - \vec{\nabla}\cdot\vec{v}_1 \underset{(1)}{\left(K_1 + h_1\tilde{p}_1\right)} - \underset{(2)}{\tilde{p}_1 \frac{\partial h_1}{\partial t}} \tag{11.13}$$

Similarly, summing (11.8) and (11.10) gives the change in total available potential energy per unit area for the one active layer, reduced-gravity model as

$$\frac{\partial}{\partial t}\left(A_1 + A_2\right) = \underset{(2)}{\tilde{p}_1 \frac{\partial h_1}{\partial t}} \; . \tag{11.14}$$

Alternately, the change in total energy per unit area in layers 1 and 2 can be found by adding (11.7) to (11.8) and (11.9) to (11.10), respectively, to get

$$\frac{\partial}{\partial t}\left(K_1 + A_1\right) = - \vec{\nabla}\cdot\vec{v}_1 \underset{(1)}{\left(K_1 + h_1\tilde{p}_1\right)} - \underset{(3)}{\left(\hat{p}_2 - \rho_1 g h_1\right)} \frac{\partial h_2}{\partial t} \tag{11.15}$$

and

$$\frac{\partial}{\partial t}\left(K_2 + A_2\right) = + \underset{(3)}{\left(\hat{p}_2 - \rho_1 g h_1\right)} \frac{\partial h_2}{\partial t} \; . \tag{11.16}$$

Finally, the change in the total energy per unit area for the one active layer, reduced-gravity model is found by summing (11.7) - (11.10) to get

$$\frac{\partial}{\partial t}\left(K_1 + A_1 + K_2 + A_2\right) = - \vec{\nabla}\cdot\vec{v}_1 \left(K_1 + h_1\tilde{p}_1\right) \tag{11.17}$$

or, from (5.52) for N=2,

$$\frac{\partial}{\partial t}\left(K_1 + A_1 + K_2 + A_2\right) = - \vec{\nabla}\cdot\vec{v}_1 \left(K_1 + h_1\hat{p}_1\right) + \hat{p}_2 \frac{\partial \eta_2}{\partial t} \; . \tag{11.18}$$

Part C, Section 12

In this section the energy relations are given for the special case of a two active layer, reduced-gravity model in the layer-layer formulation. A vertical cross section of the two active layer, reduced-gravity model is shown in Figure 15.

From (1.22) for i=1,2,3 and N=3, the available potential energy per unit area for each layer is

$$A_1 = \frac{1}{2}\,\rho_1 g\,\left[\eta_1^{\,2} - \left(\eta_2^{\,2} - 2\eta_2 H_1\right)\right]\,, \tag{12.1}$$

$$A_2 = \frac{1}{2}\,\rho_2 g\,\left[\left(\eta_2^{\,2} - 2\eta_2 H_1\right) - \left(\eta_3^{\,2} - 2\eta_3\,\left(H_1 + H_2\right)\right)\right]\,, \tag{12.2}$$

$$A_3 = \frac{1}{2}\,\rho_3 g\,\left[\eta_3^{\,2} - 2\eta_3\,\left(H_1 + H_2\right)\right]\,. \tag{12.3}$$

From (1.1) for i=1,2, and from the discussion following (5.3), the kinetic energy per unit area in layers 1,2, and 3 can be written as

$$K_1 = \frac{1}{2}\,\rho_1 h_1\,\left(u_1^{\,2} + v_1^{\,2}\right)\,, \tag{12.4}$$

$$K_2 = \frac{1}{2}\,\rho_2 h_2\,\left(u_2^{\,2} + v_2^{\,2}\right)\,, \tag{12.5}$$

$$K_3 = 0\,. \tag{12.6}$$

The total kinetic energy per unit area for the two active layer, reduced-gravity model is found by summing (12.4) - (12.6) to get

$$K_1 + K_2 + K_3 = \frac{1}{2}\,\rho_1 h_1\,\left(u_1^{\,2} + v_1^{\,2}\right) + \frac{1}{2}\,\rho_2 h_2\,\left(u_2^{\,2} + v_2^{\,2}\right)\,. \tag{12.7}$$

Similarly, the total available potential energy per unit area for the two active layer, reduced-gravity model is found by summing (12.1) - (12.3) to get

$$A_1 + A_2 + A_3 = \frac{1}{2}\,\rho_1 g\,\eta_1^{\,2} + \frac{1}{2}\,\left(\rho_2 - \rho_1\right) g\,\left(\eta_2^{\,2} - 2\eta_2 H_1\right) + \frac{1}{2}\,\left(\rho_3 - \rho_2\right) g\,\left[\eta_3^{\,2} - 2\eta_3\,\left(H_1 + H_2\right)\right]\,. \tag{12.8}$$

From (5.44) - (5.46) for i=1,2,3 and N=3, the energy relations for a two active layer, reduced-gravity model are

$$\frac{\partial K_1}{\partial t} = - \vec{\nabla}\cdot\vec{v}_1 \underset{(1)}{\left(K_1 + h_1\tilde{p}_1\right)} - \underset{(2)}{\tilde{p}_1 \frac{\partial h_1}{\partial t}} , \tag{12.9}$$

$$\frac{\partial A_1}{\partial t} = \underset{(2)}{\tilde{p}_1 \frac{\partial h_1}{\partial t}} - \underset{(3)}{\left[\left(\frac{\hat{p}_3}{\rho_3}\right)\left(\rho_2 \frac{\partial h_2}{\partial t} + \rho_3 \frac{\partial h_3}{\partial t}\right) - \left(\rho_1 g h_1\right)\left(\frac{\partial h_2}{\partial t} + \frac{\partial h_3}{\partial t}\right)\right]} , \tag{12.10}$$

$$\frac{\partial K_2}{\partial t} = - \vec{\nabla}\cdot\vec{v}_2 \underset{(4)}{\left(K_2 + h_2\tilde{p}_2\right)} - \underset{(5)}{\tilde{p}_2 \frac{\partial h_2}{\partial t}} , \tag{12.11}$$

$$\frac{\partial A_2}{\partial t} = \underset{(5)}{\tilde{p}_2 \frac{\partial h_2}{\partial t}} - \underset{(6)}{\left[\hat{p}_3 - \left(\rho_1 g h_1 + \rho_2 g h_2\right)\right]} \frac{\partial h_3}{\partial t}$$

$$+ \underset{(3)}{\left[\left(\frac{\hat{p}_3}{\rho_3}\right)\left(\rho_2 \frac{\partial h_2}{\partial t} + \rho_3 \frac{\partial h_3}{\partial t}\right) - \left(\rho_1 g h_1\right)\left(\frac{\partial h_2}{\partial t} + \frac{\partial h_3}{\partial t}\right)\right]} , \tag{12.12}$$

$$\frac{\partial K_3}{\partial t} = 0 , \tag{12.13}$$

$$\frac{\partial A_3}{\partial t} = \underset{(6)}{\left[\hat{p}_3 - \left(\rho_1 g h_1 + \rho_2 g h_2\right)\right]} \frac{\partial h_3}{\partial t} , \tag{12.14}$$

where, from (5.35) and (2.5), for i=1,2

$$\tilde{p}_1 = \rho_1 g \eta_1 - \left(\frac{\rho_1}{\rho_3}\right)\hat{p}_3 = \hat{p}_1 - \left(\frac{\rho_1}{\rho_3}\right)\hat{p}_3 , \tag{12.15}$$

$$\tilde{p}_2 = \rho_2 g \eta_1 - \left(\rho_2 - \rho_1\right) g h_1 - \left(\frac{\rho_2}{\rho_3}\right)\hat{p}_3 = \hat{p}_2 - \left(\frac{\rho_2}{\rho_3}\right)\hat{p}_3 , \tag{12.16}$$

and, from (5.20), N=3

$$\hat{p}_3 = \left(\rho_1 - \rho_3\right)gH_1 + \left(\rho_2 - \rho_3\right)gH_2 \ . \tag{12.17}$$

The energy relations, (12.9) - (12.14), are shown in the block diagram in Figure 16.

Summing (12.9), (12.11), and (12.13) gives the change in the total kinetic energy per unit area for the two active layer, reduced-gravity model as

$$\frac{\partial}{\partial t}\left(K_1 + K_2 + K_3\right) = \underset{(1)}{-\vec{\nabla}\cdot\vec{v}_1\left(K_1 + h_1\tilde{p}_1\right)} \underset{(2)}{- \tilde{p}_1\frac{\partial h_1}{\partial t}}$$

$$\underset{(4)}{- \vec{\nabla}\cdot\vec{v}_2\left(K_2 + h_2\tilde{p}_2\right)} \underset{(5)}{- \tilde{p}_2\frac{\partial h_2}{\partial t}} \ . \tag{12.18}$$

Similarly, summing (12.10), (12.12) and (12.14) gives the change in the total available potential energy per unit area for the two active layer, reduced-gravity model as

$$\frac{\partial}{\partial t}\left(A_1 + A_2 + A_3\right) = \underset{(2)}{\tilde{p}_1\frac{\partial h_1}{\partial t}} + \underset{(5)}{\tilde{p}_2\frac{\partial h_2}{\partial t}} \ . \tag{12.19}$$

Alternatively, the change in total energy per unit area in layers 1,2, and 3 can be found by adding (12.9) to (12.10), (12.11) to (12.12), and (12.13) to (12.14), respectively, to get

$$\frac{\partial}{\partial t}\left(K_1 + A_1\right) = \underset{(1)}{-\vec{\nabla}\cdot\vec{v}_1\left(K_1 + h_1\tilde{p}_1\right)}$$

$$\underset{(3)}{- \left[\left(\frac{\hat{p}_3}{\rho_3}\right)\left(\rho_2\frac{\partial h_2}{\partial t} + \rho_3\frac{\partial h_3}{\partial t}\right) - \left(\rho_1 g h_1\right)\left(\frac{\partial h_2}{\partial t} + \frac{\partial h_3}{\partial t}\right)\right]} \ , \tag{12.20}$$

$$\frac{\partial}{\partial t}\left(K_2 + A_2\right) = -\vec{\nabla}\cdot\vec{v}_2\left(K_2 + h_2\tilde{p}_2\right) - \left[\hat{p}_3 - \left(\rho_1 g h_1 + \rho_2 g h_2\right)\right]\frac{\partial h_3}{\partial t}$$

(4) (6)

$$+\left[\left(\frac{\hat{p}_3}{\rho_3}\right)\left(\rho_2\frac{\partial h_2}{\partial t} + \rho_3\frac{\partial h_3}{\partial t}\right) - \left(\rho_1 g h_1\right)\left(\frac{\partial h_2}{\partial t} + \frac{\partial h_3}{\partial t}\right)\right], \qquad (12.21)$$

(3)

$$\frac{\partial}{\partial t}\left(K_3 + A_3\right) = \left[\hat{p}_3 - \left(\rho_1 g h_1 + \rho_2 g h_2\right)\right]\frac{\partial h_3}{\partial t}. \qquad (12.22)$$

(6)

Finally, the change in the total energy per unit area for the two active layer, reduced-gravity model is found by summing (12.9) - (12.14) to get

$$\frac{\partial}{\partial t}\left(K_1 + A_1 + K_2 + A_2 + K_3 + A_3\right) = -\vec{\nabla}\cdot\vec{v}_1\left(K_1 + h_1\tilde{p}_1\right)$$

(1)

$$-\vec{\nabla}\cdot\vec{v}_2\left(K_2 + h_2\tilde{p}_2\right) \qquad (12.23)$$

(4)

or, from (5.52) for N=3,

$$\frac{\partial}{\partial t}\left(K_1 + A_1 + K_2 + A_2 + K_3 + A_3\right) = -\vec{\nabla}\cdot\vec{v}_1\left(K_1 + h_1\hat{p}_1\right)$$

$$-\vec{\nabla}\cdot\vec{v}_2\left(K_2 + h_2\hat{p}_2\right) + \hat{p}_3\frac{\partial\eta_3}{\partial t}. \qquad (12.24)$$

Part D, Section 13

In this section the energy specifications and relations are given for the special case of a one-layer, finite depth, model in the layer-interface formulation. The reader should be familiar with section B.3 where the layer-interface formulation is discussed in detail for the finite depth model. A vertical cross-section of the one-layer model is shown in Figure 7. From (1.1), for i=1 and N=1, the kinetic energy per unit area for the layer is

$$K_1 = \frac{1}{2} h_1 \rho_1 \left( u_1^2 + v_1^2 \right) . \tag{13.1}$$

For the interface formulation, the available potential energy per unit area is found in (3.4). For i=1 and N=1, this relation is

$$\tilde{A}_1 = \frac{1}{2} \rho_1 g \eta_1^2 . \tag{13.2}$$

From (3.26) and (3.27), for i=1 and N=1, the energy relations for the one layer model in layer-interface formulation are

$$\frac{\partial K_1}{\partial t} = - \vec{\nabla} \cdot \vec{v}_1 \underbrace{\left( K_1 + h_1 \hat{p}_1 \right)}_{(1)} - \underbrace{\hat{p}_1 \frac{\partial \eta_1}{\partial t}}_{(2)} \tag{13.3}$$

and

$$\frac{\partial \tilde{A}_1}{\partial t} = \underbrace{\hat{p}_1 \frac{\partial \eta_1}{\partial t}}_{(2)} . \tag{13.4}$$

where, from (2.5) and Figure 7,

$$\hat{p}_1 = \rho_1 g \eta_1 = \rho_1 g \left( h_1 - H_1 \right) . \tag{13.5}$$

The energy transfers in (13.3)-(13.4) are shown in the block diagram in Figure 17. Adding (13.3) and (13.4) gives

$$\frac{\partial}{\partial t} \left( K_1 + \tilde{A}_1 \right) = \underbrace{- \vec{\nabla} \cdot \vec{v}_1 \left( K_1 + h_1 \hat{p}_1 \right)}_{(1)} , \tag{13.6}$$

as expected from (3.33) for N=1.

Eliminate $\eta_1$ from (13.3) and (13.4) by using (13.5). The results are

$$\frac{\partial K_1}{\partial t} = - \vec{\nabla}\cdot\vec{v} \underset{(1)}{\left(K_1 + h_1\hat{p}_1\right)} - \underset{(2)}{\hat{p}_1 \frac{\partial h_1}{\partial t}} \tag{13.7}$$

$$\frac{\partial \tilde{A}_1}{\partial t} = \underset{(2)}{\hat{p}_1 \frac{\partial h_1}{\partial t}} . \tag{13.8}$$

Note that the energy relations for the one layer model in the layer-layer formulation, (8.3) - (8.4), are identical to those in the layer-interface formulation, (13.7) - (13.8). Furthermore, the total kinetic energy, (8.1) and (13.1), and the total available potential energy, (8.2) and (13.2), are identical for the two formulations as expected.

Part D, Section 14

In this section the energy relations are given for the special case of a two-layer, finite depth model in the layer-interface formulation. A vertical cross section of the two layer model is shown in Figure 9. From (1.1), for i=1,2 and N=2, the kinetic energy per unit area in layers 1 and 2 can be written as

$$K_1 = \frac{1}{2} h_1 \rho_1 \left(u_1^2 + v_1^2\right) \tag{14.1}$$

and

$$K_2 = \frac{1}{2} h_2 \rho_2 \left(u_2^2 + v_2^2\right) \tag{14.2}$$

From (3.4), the available potential energy per unit area in the interface formulation for layers 1 and 2 is expressed as

$$\tilde{A}_1 = \frac{1}{2} \rho_1 g \eta_1^2 \tag{14.3}$$

and

$$\tilde{A}_2 = \frac{1}{2} \left(\rho_2 - \rho_1\right) g \left(\eta_2^2 - 2\eta_2 H_1\right) , \tag{14.4}$$

The total kinetic energy per unit area for this case is found by adding (14.1) and (14.2) to get

$$\left(K_1 + K_2\right) = \frac{1}{2} h_1 \rho_1 \left(u_1^2 + v_1^2\right) + \frac{1}{2} h_2 \rho_2 \left(u_2^2 + v_2^2\right) . \tag{14.5}$$

Similarly the total available potential energy per unit area for this case is found by adding (14.3) and (14.4) to find

$$\tilde{A}_1 + \tilde{A}_2 = \frac{1}{2} \rho_1 g \eta_1^2 + \frac{1}{2} \left(\rho_2 - \rho_1\right) g \left(\eta_2^2 - 2\eta_2 H_1\right) . \tag{14.6}$$

Note that the total kinetic energy for the two-layer case is the same in the layer-layer formulation, (9.5), and in the layer-interface formulation. The total available potential energy is the same in both formulations, (9.6) vs (14.6).

From (3.26) and (3.27), for i=1,2 and N=2, the energy relations for a two-layer model in the layer-interface formulation are

$$\frac{\partial \tilde{A}_1}{\partial t} = \underset{(1)}{\hat{p}_1 \frac{\partial \eta_1}{\partial t}} , \tag{14.7}$$

$$\frac{\partial K_1}{\partial t} = - \vec{\nabla}\cdot\vec{v}_1 \underset{(2)}{\left(K_1 + h_1\hat{p}_1\right)} - \underset{(1)}{\hat{p}_1 \frac{\partial \eta_1}{\partial t}} + \underset{(3)}{\hat{p}_1 \frac{\partial \eta_2}{\partial t}} , \tag{14.8}$$

$$\frac{\partial \tilde{A}_2}{\partial t} = \underset{(4)}{\hat{p}_2 \frac{\partial \eta_2}{\partial t}} - \underset{(3)}{\hat{p}_1 \frac{\partial \eta_2}{\partial t}} , \tag{14.9}$$

$$\frac{\partial K_2}{\partial t} = - \vec{\nabla}\cdot\vec{v}_2 \underset{(5)}{\left(K_2 + h_2\hat{p}_2\right)} - \underset{(4)}{\hat{p}_2 \frac{\partial \eta_2}{\partial t}} , \tag{14.10}$$

where the pressures, $\hat{p}_1$ and $\hat{p}_2$, are given from (2.5) with i=1,2 as

$$\hat{p}_1 = \rho_1 g \eta_1 \tag{14.11}$$

$$\hat{p}_2 = \rho_2 g \eta_1 - \left(\rho_2 - \rho_1\right) g h_1 . \tag{14.12}$$

The energy transfers in (14.7) - (14.10), are shown in Figure 18.

Summing (14.7) and (14.9) gives the change in total available potential energy per unit area for this case that is

$$\frac{\partial}{\partial t}\left(\tilde{A}_1 + \tilde{A}_2\right) = \underset{(1)}{\hat{p}_1 \frac{\partial \eta_1}{\partial t}} - \underset{(3)}{\hat{p}_1 \frac{\partial \eta_2}{\partial t}} + \underset{(4)}{\hat{p}_2 \frac{\partial \eta_2}{\partial t}} . \tag{14.13}$$

From Figure 9, the relations between interface deviation, $\eta_i$, and the layer depth, $h_i$, for i=1 and 2 are expressed as

$$h_1 = \eta_1 + H_1 - \eta_2 \tag{14.14}$$

and

$$h_2 = H_2 + \eta_2 \ . \tag{14.15}$$

Substituting (14.14) - (14.15) into (14.13) yields

$$\frac{\partial}{\partial t}\left(\tilde{A}_1 + \tilde{A}_2\right) = \hat{p}_1 \frac{\partial h_1}{\partial t} + \hat{p}_2 \frac{\partial h_2}{\partial t} \ . \tag{14.16}$$

A comparison of (14.16) and (9.14) reveals that the change in total available potential energy per unit area for the two-layer case is the same for both the layer-layer and the layer-interface formulations.

The energetics for a two-layer, finite depth model have been derived by Hurlburt and Thompson (1982). They specify the kinetic energy per unit area for each layer, but use the total available potential energy per unit area for the system. Thus, the question of which formulation to use for the available potential energy does not arise. The relations derived here are equivalent to those in their paper with three differences: (1) They use $\rho$ instead of $\rho_1$ and $\rho_2$ when these values appear alone, (2) the relations derived here have no source or sink terms included, and (3) the relations derived here ((14.8), (14.10), and (14.13)) contain extra terms that vanish if averaged over the entire model region.

Holland and Lin (1975) also have examined the energetics for a two-layer, finite depth model. They specify the kinetic energy per unit area for each layer and the total available potential energy per unit area for the system. Then they average these quantities over the entire basin. However, they use a rigid top ($\eta_1 = 0$) with their model so the area-averaged, total available potential energy is simply the area-averaged, available potential energy associated with the interface between the two layers.

Summing (14.8) and (14.10) yields the change in total kinetic energy per unit area for this case as

$$\frac{\partial}{\partial t}\left(K_1 + K_2\right) = \underset{(2)}{-\vec{\nabla}\cdot\vec{v}_1}\left(K_1 + h_1\hat{p}_1\right) \underset{(1)}{-\hat{p}_1 \frac{\partial \eta_1}{\partial t}} \underset{(3)}{+\hat{p}_1 \frac{\partial \eta_2}{\partial t}}$$

$$\underset{(5)}{-\vec{\nabla}\cdot\vec{v}_2}\left(K_2 + h_2\hat{p}_2\right) \underset{(4)}{-\hat{p}_2 \frac{\partial \eta_2}{\partial t}} \ . \tag{14.17}$$

Substituting (14.14) - (14.15) into (14.17) gives

$$\frac{\partial}{\partial t}\left(K_1 + K_2\right) = -\vec{\nabla}\cdot\vec{v}_1\left(K_1 + h_1\hat{p}_1\right) - \hat{p}_1\frac{\partial h_1}{\partial t}$$

$$-\vec{\nabla}\cdot\vec{v}_2\left(K_2 + h_2\hat{p}_2\right) - \hat{p}_2\frac{\partial h_2}{\partial t}\,, \qquad (14.18)$$

which is the same as (9.13), the change in total kinetic energy per unit area in the layer-layer formulation for the two-layer case.

The change in total energy per unit area for the system is found by summing (14.7) - (14.10) to get

$$\frac{\partial}{\partial t}\left(\tilde{A}_1 + K_1 + \tilde{A}_2 + K_2\right) = -\vec{\nabla}\cdot\vec{v}_1\left(K_1 + h_1\hat{p}_1\right) - \vec{\nabla}\cdot\vec{v}_2\left(K_2 + h_2\hat{p}_2\right)\,, \qquad (14.19)$$

$$(2) \qquad (5)$$

as expected from (3.33). Note that (14.19) is identical to (9.17), which shows that the change in total energy per unit area for the two-layer case of the finite depth model does not depend on the type of formulation for available potential energy.

Part D, Section 15

In this section the energy specifications and relations are given for the special case of a three-layer, finite depth model in the layer-interface formulation. A vertical cross section of the three-layer model is shown in Figure 11. From (1.1), for i=1,2,3 and N=3, the kinetic energy per unit area in layers 1, 2, and 3 can be written as

$$K_1 = \frac{1}{2} h_1 \rho_1 \left(u_1^2 + v_1^2\right) , \tag{15.1}$$

$$K_2 = \frac{1}{2} h_2 \rho_2 \left(u_2^2 + v_2^2\right) , \tag{15.2}$$

$$K_3 = \frac{1}{2} h_3 \rho_3 \left(u_3^2 + v_3^2\right) . \tag{15.3}$$

The total kinetic energy per unit area for this case is found by summing (15.1) - (15.3) to get

$$K_1 + K_2 + K_3 = \frac{1}{2} h_1 \rho_1 \left(u_1^2 + v_1^2\right) + \frac{1}{2} h_2 \rho_2 \left(u_2^2 + v_2^2\right) + \frac{1}{2} h_3 \rho_3 \left(u_3^2 + v_3^2\right) . \tag{15.4}$$

From (3.4), for i=1,2, and 3 and N=3, the available potential energy per unit area for layers 1, 2 and 3 can be expressed as

$$\tilde{A}_1 = \frac{1}{2} \rho_1 g \eta_1^2 , \tag{15.5}$$

$$\tilde{A}_2 = \frac{1}{2} \left(\rho_2 - \rho_1\right) g \left(\eta_2^2 - 2\eta_2 H_1\right) , \tag{15.6}$$

$$\tilde{A}_3 = \frac{1}{2} \left(\rho_3 - \rho_2\right) g \left[\eta_3^2 - 2\eta_3 \left(H_1 + H_2\right)\right] . \tag{15.7}$$

The total available potential energy per unit area for this case is found by summing (15.5) - (15.7) to get

$$\tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3 = \frac{1}{2} \rho_1 g \eta_1^2 + \frac{1}{2} \left(\rho_2 - \rho_1\right) g \left(\eta_2^2 - 2\eta_2 H_1\right) + \frac{1}{2} \left(\rho_3 - \rho_2\right) g \left[\eta_3^2 - 2\eta_3 \left(H_1 + H_2\right)\right] . \tag{15.8}$$

Note that the total kinetic energy is the same for the three-layer model in the layer-layer formulation, (10.7), and the layer-interface formulation, (15.4). Similarly, the total available potential energy per unit area is the same for both formulations, (10.8) and (15.8).

From (3.26) and (3.27), for i=1, 2 and 3 and N=3, the energy relations for a three-layer, finite depth model in the layer-interface formulation are

$$\frac{\partial \tilde{A}_1}{\partial t} = \underset{(1)}{\hat{p}_1} \frac{\partial \eta_1}{\partial t} , \tag{15.9}$$

$$\frac{\partial K_1}{\partial t} = - \vec{\nabla}\cdot\vec{v}_1 \underset{(2)}{\left(K_1 + h_1\hat{p}_1\right)} - \underset{(1)}{\hat{p}_1} \frac{\partial \eta_1}{\partial t} + \underset{(3)}{\hat{p}_1} \frac{\partial \eta_2}{\partial t} , \tag{15.10}$$

$$\frac{\partial \tilde{A}_2}{\partial t} = \underset{(4)}{\hat{p}_2} \frac{\partial \eta_2}{\partial t} - \underset{(3)}{\hat{p}_1} \frac{\partial \eta_2}{\partial t} , \tag{15.11}$$

$$\frac{\partial K_2}{\partial t} = - \vec{\nabla}\cdot\vec{v}_2 \underset{(5)}{\left(K_2 + h_2\hat{p}_2\right)} - \underset{(4)}{\hat{p}_2} \frac{\partial \eta_2}{\partial t} + \underset{(6)}{\hat{p}_2} \frac{\partial \eta_3}{\partial t} , \tag{15.12}$$

$$\frac{\partial \tilde{A}_3}{\partial t} = \underset{(7)}{\hat{p}_3} \frac{\partial \eta_3}{\partial t} - \underset{(6)}{\hat{p}_2} \frac{\partial \eta_3}{\partial t} , \tag{15.13}$$

$$\frac{\partial K_3}{\partial t} = - \vec{\nabla}\cdot\vec{v}_3 \underset{(8)}{\left(K_3 + h_3\hat{p}_3\right)} - \underset{(7)}{\hat{p}_3} \frac{\partial \eta_3}{\partial t} , \tag{15.14}$$

where the pressure, $\hat{p}_i$, is given in (2.5) for i=1, 2 and 3 as

$$\hat{p}_1 = \rho_1 g \eta_1 , \tag{15.15}$$

$$\hat{p}_2 = \rho_2 g \eta_1 - \left(\rho_2 - \rho_1\right) g h_1 , \tag{15.16}$$

and

$$\hat{p}_3 = \rho_3 g\eta_1 - \left(\rho_3 - \rho_1\right)gh_1 - \left(\rho_3 - \rho_2\right)gh_2 \ . \tag{15.17}$$

These energy relations, (15.9) - (15.14), are shown in the block diagram in Figure 19.

Summing (15.9), (15.11) and (15.13) gives the change in the total available potential energy per unit area for this case that is

$$\frac{\partial}{\partial t}\left(\tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3\right) = \hat{p}_1\left(\frac{\partial \eta_1}{\partial t} - \frac{\partial \eta_2}{\partial t}\right) + \hat{p}_2\left(\frac{\partial \eta_2}{\partial t} - \frac{\partial \eta_3}{\partial t}\right) + \hat{p}_3\frac{\partial \eta_3}{\partial t} \ . \tag{15.18}$$

From Figure 11, the relations between interface deviation, $\eta_i$, and the layer depth, $h_i$, for i=1,2, and 3 are expressed as

$$h_1 = \eta_1 + H_1 - \eta_2 \ , \tag{15.19}$$

$$h_2 = \eta_2 + H_2 - \eta_3 \ , \tag{15.20}$$

$$h_3 = \eta_3 + H_3 \ . \tag{15.21}$$

Substituting (15.19) - (15.21) into (15.18) yields

$$\frac{\partial}{\partial t}\left(\tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3\right) = \hat{p}_1\frac{\partial h_1}{\partial t} + \hat{p}_2\frac{\partial h_2}{\partial t} + \hat{p}_3\frac{\partial h_3}{\partial t} \ . \tag{15.22}$$

A comparison of (15.22) and (10.19) reveals that the change in total available potential energy per unit area for the three-layer case of the finite depth model is the same for the layer-layer and layer-interface formulations.

The change w.r.t. time in the total kinetic energy per unit area is found by adding (15.10), (15.12), and (15.14). The result is

$$\frac{\partial}{\partial t}\left(K_1 + K_2 + K_3\right) = \underset{(2)}{-\vec{\nabla}\cdot\vec{v}_1\left(K_1 + h_1\hat{p}_1\right)} \underset{(1)}{- \hat{p}_1\frac{\partial \eta_1}{\partial t}} \underset{(3)}{+ \hat{p}_1\frac{\partial \eta_2}{\partial t}}$$

$$- \vec{\nabla}\cdot\vec{v}_2 \underset{(5)}{\left(K_2 + h_2\hat{p}_2\right)} - \hat{p}_2 \underset{(4)}{\frac{\partial \eta_2}{\partial t}} + \hat{p}_2 \underset{(6)}{\frac{\partial \eta_3}{\partial t}}$$

$$- \vec{\nabla}\cdot\vec{v}_3 \underset{(8)}{\left(K_3 + h_3\hat{p}_3\right)} - \hat{p}_3 \underset{(7)}{\frac{\partial \eta_3}{\partial t}} . \qquad (15.23)$$

Substituting (15.19) - (15.21) into (15.23) yields

$$\frac{\partial}{\partial t}\left(K_1 + K_2 + K_3\right) = - \vec{\nabla}\cdot\vec{v}_1 \left(K_1 + h_1\hat{p}_1\right) - \hat{p}_1 \frac{\partial h_1}{\partial t}$$

$$- \vec{\nabla}\cdot\vec{v}_2 \left(K_2 + h_2\hat{p}_2\right) - \hat{p}_2 \frac{\partial h_2}{\partial t}$$

$$- \vec{\nabla}\cdot\vec{v}_3 \left(K_3 + h_3\hat{p}_3\right) - \hat{p}_3 \frac{\partial h_3}{\partial t} , \qquad (15.24)$$

which is the same as (10.18), the change in total kinetic energy per unit area in the layer-layer formulation for the three-layer, finite depth model.

The change in total energy per unit area for the system is found by summing (15.9) - (15.14) to get

$$\frac{\partial}{\partial t}\left(\tilde{A}_1 + K_1 + \tilde{A}_2 + K_2 + \tilde{A}_3 + K_3\right) = - \vec{\nabla}\cdot\vec{v}_1 \underset{(2)}{\left(K_1 + h_1\hat{p}_1\right)}$$

$$- \vec{\nabla}\cdot\vec{v}_2 \underset{(5)}{\left(K_2 + h_2\hat{p}_2\right)} - \vec{\nabla}\cdot\vec{v}_3 \underset{(8)}{\left(K_3 + h_3\hat{p}_3\right)} , \qquad (15.25)$$

as expected from (3.33) for N=3. Note that (15.25) is identical to (10.23), which shows that the change in total energy per unit area for the three-layer, finite depth model does not depend on the type of formulation.

Part D, Section 16

In this section, the energy specifications and relations are given for the special case of a one active layer, reduced-gravity model in the layer-interface formulation. The reader should be familiar with section B.6 where the layer-interface formulation for the reduced-gravity model is discussed in detail. A vertical cross-section of the one active layer, reduced-gravity model is shown in Figure 13. From (1.1), for i=1, and from the discussion following (5.3), the kinetic energy per unit area in layers 1 and 2 can be written as

$$K_1 = \frac{1}{2} h_1 \rho_1 \left(u_1^2 + v_1^2\right) \tag{16.1}$$

and

$$K_2 = 0 \ . \tag{16.2}$$

The total kinetic energy per unit area for the layers in the reduced-gravity model is found by adding (16.1) and (16.2) to get

$$K_1 + K_2 = \frac{1}{2} h_1 \rho_1 \left(u_1^2 + v_1^2\right) \tag{16.3}$$

From (3.4) for i=1,2 and N=2.

$$\tilde{A}_1 = \frac{1}{2} \rho_1 g \eta_1^2 \tag{16.4}$$

$$\tilde{A}_2 = \frac{1}{2} \left(\rho_2 - \rho_1\right) g \left(\eta_2^2 - 2\eta_2 H_1\right) \ . \tag{16.5}$$

The total available potential energy per unit area for this case is found by adding (16.4) and (16.5) to obtain

$$\tilde{A}_1 + \tilde{A}_2 = \frac{1}{2} \rho_1 g \eta_1^2 + \frac{1}{2} \left(\rho_2 - \rho_1\right) g \left(\eta_2^2 - 2\eta_2 H_1\right) \ . \tag{16.6}$$

Note that the total kinetic energy per unit area is the same for the one active layer, reduced-gravity model in the layer-layer formulation, (11.5), and the layer-interface formulation, (16.3). The total available potential energy per unit area for this case is the same for both formulations, (11.6) and (16.6).

From (6.21) - (6.23), for i=1,2 and N=2, the energy relations for a one active layer, reduced-gravity model in the layer-interface formulation are

$$\frac{\partial \tilde{A}_1}{\partial t} = \underset{(1)}{\tilde{p}_1 \frac{\partial \eta_1}{\partial t}} - \underset{(2)}{\left(\frac{\hat{p}_2}{\rho_2}\right)\left(\rho_2 - \rho_1\right)\frac{\partial \eta_2}{\partial t}}, \tag{16.7}$$

$$\frac{\partial K_1}{\partial t} = \underset{(3)}{-\vec{\nabla}\cdot\vec{v}_1\left(K_1 + h_1\tilde{p}_1\right)} \underset{(1)}{-\tilde{p}_1\frac{\partial \eta_1}{\partial t}} \underset{(4)}{+\tilde{p}_1\frac{\partial \eta_2}{\partial t}}, \tag{16.8}$$

$$\frac{\partial \tilde{A}_2}{\partial t} = \underset{(4)}{-\tilde{p}_1\frac{\partial \eta_2}{\partial t}} + \underset{(2)}{\left(\frac{\hat{p}_2}{\rho_2}\right)\left(\rho_2 - \rho_1\right)\frac{\partial \eta_2}{\partial t}}, \tag{16.9}$$

$$\frac{\partial K_2}{\partial t} = 0, \tag{16.10}$$

where, from (6.11) and (2.5), for i=1,

$$\tilde{p}_1 = \rho_1 g \eta_1 - \left(\frac{\rho_1}{\rho_2}\right)\hat{p}_2 = \hat{p}_1 - \left(\frac{\rho_1}{\rho_2}\right)\hat{p}_2 \tag{16.11}$$

and, from (6.4) and N=2,

$$\hat{p}_2 = \left(\rho_1 - \rho_2\right) g H_1 \tag{16.12}$$

These energy relations, (16.7) - (16.10), are shown in the block diagram in Figure 20.

Summing (16.8) and (16.10) gives the change in the total kinetic energy per unit area for the one active layer, reduced-gravity model in the layer-interface formulation as

$$\frac{\partial}{\partial t}\left(K_1 + K_2\right) = \underset{(3)}{-\vec{\nabla}\cdot\vec{v}_1\left(K_1 + h_1\tilde{p}_1\right)} \underset{(1)}{-\tilde{p}_1\frac{\partial \eta_1}{\partial t}} \underset{(4)}{+\tilde{p}_1\frac{\partial \eta_2}{\partial t}} \tag{16.13}$$

From Figure 13, the relation between interface deviation, $\eta_i$, and depth, $h_i$, for i=1,2 is

$$h_1 = \eta_1 + H_1 - \eta_2 \ . \tag{16.14}$$

Using (16.14) in (16.13) yields

$$\frac{\partial}{\partial t}\left(K_1 + K_2\right) = - \vec{\nabla}\cdot\vec{v}_1 \left(K_1 + h_1\tilde{p}_1\right) - \tilde{p}_1 \frac{\partial h_1}{\partial t} \ , \tag{16.15}$$

which is the same as (11.13), the change in total kinetic energy per unit area in the one active layer, reduced-gravity model in the layer-layer formulation.

Summing (16.7) and (16.9) gives the change in the total available potential energy per unit area for this case, which is

$$\frac{\partial}{\partial t}\left(\tilde{A}_1 + \tilde{A}_2\right) = \tilde{p}_1 \left(\frac{\partial \eta_1}{\partial t} - \frac{\partial \eta_2}{\partial t}\right) \ . \tag{16.16}$$

Substituting (16.14) into (16.16) yields

$$\frac{\partial}{\partial t}\left(\tilde{A}_1 + \tilde{A}_2\right) = \tilde{p}_1 \frac{\partial h_1}{\partial t} \ . \tag{16.17}$$

A comparison of (16.17) and (11.14) shows that the change in total available potential energy per unit area for the one active layer, reduced-gravity model does not differ between the layer-layer and the layer-interface formulations.

The change in the total energy per unit area for this case is found by summing (16.7) - (16.10) to find

$$\frac{\partial}{\partial t}\left(\tilde{A}_1 + K_1 + \tilde{A}_2 + K_2\right) = - \vec{\nabla}\cdot\vec{v}_1 \left(K_1 + h_1\tilde{p}_1\right) \tag{16.18}$$



or, from (6.28)

$$\frac{\partial}{\partial t}\left(\tilde{A}_1 + K_1 + \tilde{A}_2 + K_2\right) = - \vec{\nabla}\cdot\vec{v}_1 \left(K_1 + h_1\hat{p}_1\right) + \hat{p}_2 \frac{\partial \eta_2}{\partial t} \ . \tag{16.19}$$

Note that (16.19) and (11.18) are identical so that the change in total energy per unit area does not depend on the type of formulation: layer-layer or layer-interface.

Part C, Section 17

In this section, the energy specifications and relations are given for the special case of a two active layer, reduced-gravity model in layer-interface formulation. A vertical cross-section of the two active layer, reduced-gravity model is shown in Figure 15. From (1.1) for i=1,2 and from the discussion following (5.3) for N=3, the kinetic energy per unit area in layers 1,2 and 3 can be written as

$$K_1 = \frac{1}{2} h_1 \rho_1 \left(u_1^2 + v_1^2\right) , \tag{17.1}$$

$$K_2 = \frac{1}{2} h_2 \rho_2 \left(u_2^2 + v_2^2\right) , \tag{17.2}$$

$$K_3 = 0 . \tag{17.3}$$

The total kinetic energy per unit area for all layers in this reduced-gravity model is found by summing (17.1) - (17.3) to get

$$K_1 + K_2 + K_3 = \frac{1}{2} h_1 \rho_1 \left(u_1^2 + v_1^2\right) + \frac{1}{2} h_2 \rho_2 \left(u_2^2 + v_2^2\right) . \tag{17.4}$$

From (3.4) for i=1,2,3 and N=3, the available potential energy in the interface formulation for layers 1-3 can be written as

$$\tilde{A}_1 = \frac{1}{2} \rho_1 g \eta_1^2 , \tag{17.5}$$

$$\tilde{A}_2 = \frac{1}{2} \left(\rho_2 - \rho_1\right) g \left(\eta_2^2 - 2\eta_2 H_1\right) , \tag{17.6}$$

$$\tilde{A}_3 = \frac{1}{2} \left(\rho_3 - \rho_2\right) g \left[\eta_3^2 - 2\eta_3 \left(H_1 + H_2\right)\right] . \tag{17.7}$$

The total available potential energy for this case is found by summing (17.5) - (17.7) to obtain

$$\tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3 = \frac{1}{2} \rho_1 g \eta_1^2 + \frac{1}{2} \left(\rho_2 - \rho_1\right) g \left(\eta_2^2 - 2\eta_2 H_1\right) + \frac{1}{2} \left(\rho_3 - \rho_2\right) g \left[\eta_3^2 - 2\eta_3 \left(H_1 + H_2\right)\right] . \tag{17.8}$$

Note that the total kinetic energy per unit area is the same for the two active layer, reduced-gravity model in the layer-layer formulation, (12.7),

and for the layer-interface formulation, (17.4). The total available potential energy per unit area for this case is the same for both formulations, (12.8) and (17.8).

From (6.21) - (6.23) for N=3, the energy relations for a two active layer, reduced-gravity model in the layer-interface formulation are

$$\frac{\partial \tilde{A}_1}{\partial t} = \underset{(1)}{\tilde{p}_1 \frac{\partial \eta_1}{\partial t}} - \left(\frac{\hat{p}_3}{\rho_3}\right) \underset{(2)}{\left[\left(\rho_2 - \rho_1\right) \frac{\partial \eta_2}{\partial t} + \left(\rho_3 - \rho_2\right) \frac{\partial \eta_3}{\partial t}\right]} , \tag{17.9}$$

$$\frac{\partial K_1}{\partial t} = \underset{(3)}{- \vec{\nabla}\cdot\vec{v}_1 \left(K_1 + h_1 \tilde{p}_1\right)} \underset{(1)}{- \tilde{p}_1 \frac{\partial \eta_1}{\partial t}} \underset{(4)}{+ \tilde{p}_1 \frac{\partial \eta_2}{\partial t}} , \tag{17.10}$$

$$\frac{\partial \tilde{A}_2}{\partial t} = \underset{(5)}{\tilde{p}_2 \frac{\partial \eta_2}{\partial t}} \underset{(4)}{- \tilde{p}_1 \frac{\partial \eta_2}{\partial t}} + \left(\frac{\hat{p}_3}{\rho_3}\right) \underset{(2)}{\left[\left(\rho_2 - \rho_1\right) \frac{\partial \eta_2}{\partial t} + \left(\rho_3 - \rho_2\right) \frac{\partial \eta_3}{\partial t}\right]}$$

$$- \left(\frac{\hat{p}_3}{\rho_3}\right) \underset{(6)}{\left[\left(\rho_3 - \rho_2\right) \frac{\partial \eta_3}{\partial t}\right]} , \tag{17.11}$$

$$\frac{\partial K_2}{\partial t} = \underset{(7)}{- \vec{\nabla}\cdot\vec{v}_2 \left(K_2 + h_2 \tilde{p}_2\right)} \underset{(5)}{- \tilde{p}_2 \frac{\partial \eta_2}{\partial t}} \underset{(8)}{+ \tilde{p}_2 \frac{\partial \eta_3}{\partial t}} , \tag{17.12}$$

$$\frac{\partial \tilde{A}_3}{\partial t} = \underset{(8)}{- \tilde{p}_2 \frac{\partial \eta_3}{\partial t}} + \left(\frac{\hat{p}_3}{\rho_3}\right) \underset{(6)}{\left[\left(\rho_3 - \rho_2\right) \frac{\partial \eta_3}{\partial t}\right]} , \tag{17.13}$$

$$\frac{\partial K_3}{\partial t} = 0 , \tag{17.14}$$

where, from (6.11) and (2.5), for i=1 and 2,

$$\tilde{p}_1 = \rho_1 g\eta_1 - \left(\frac{\rho_1}{\rho_3}\right)\hat{p}_3 = \hat{p}_1 - \left(\frac{\rho_1}{\rho_3}\right)\hat{p}_3 \ , \tag{17.15}$$

$$\tilde{p}_2 = \rho_2 g\eta_1 - \left(\rho_2 - \rho_1\right)gh_1 - \left(\frac{\rho_1}{\rho_3}\right)\hat{p}_3 = \hat{p}_2 - \left(\frac{\rho_2}{\rho_3}\right)\hat{p}_3 \ , \tag{17.16}$$

and, from (6.4), for N=3,

$$\hat{p}_3 = \left(\rho_1 - \rho_3\right)gH_1 + \left(\rho_2 - \rho_3\right)gH_2 \ . \tag{17.17}$$

These energy transfers in (17.9)-(17.14) are shown in the block diagram in Figure 21.

Summing (17.10), (17.12), and (17.14) gives the change in the total kinetic energy per unit area for the two active layer, reduced-gravity model in the layer-interface formulation as

$$\frac{\partial}{\partial t}\left(K_1 + K_2 + K_3\right) = \underset{(3)}{-\vec{\nabla}\cdot\vec{v}_1\left(K_1 + h_1\tilde{p}_1\right)} \underset{(1)}{- \tilde{p}_1\frac{\partial\eta_1}{\partial t}} \underset{(4)}{+ \tilde{p}_1\frac{\partial\eta_2}{\partial t}}$$

$$\underset{(7)}{- \vec{\nabla}\cdot\vec{v}_2\left(K_2 + h_2\tilde{p}_2\right)} \underset{(5)}{- \tilde{p}_2\frac{\partial\eta_2}{\partial t}} \underset{(6)}{+ \tilde{p}_2\frac{\partial\eta_3}{\partial t}} \ . \tag{17.18}$$

From Figure 15, the relation between interface deviation, $\eta_i$, and layer depth, $h_i$, can be written for layers 1 and 2 as

$$h_1 = \eta_1 + H_1 - \eta_2 \tag{17.19}$$

and

$$h_2 = \eta_2 + H_2 - \eta_3 \ . \tag{17.20}$$

Using (17.19) - (17.20) in (17.18) yields

$$\frac{\partial}{\partial t}\left(K_1 + K_2 + K_3\right) = -\vec{\nabla}\cdot\vec{v}_1\left(K_1 + h_1\tilde{p}_1\right) - \tilde{p}_1\left(\frac{\partial h_1}{\partial t}\right)$$

$$-\vec{\nabla}\cdot\vec{v}_2\left(K_2 + h_2\tilde{p}_2\right) - \tilde{p}_2\left(\frac{\partial h_2}{\partial t}\right), \qquad (17.21)$$

which is the same as (12.18), the change in total kinetic energy per unit area for the two active layer, reduced-gravity model in the layer-layer formulation.

Summing (17.9), (17.11) and (17.13) gives the change in the total available potential energy per unit area for this case, which is

$$\frac{\partial}{\partial t}\left(\tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3\right) = \underset{(1)}{\tilde{p}_1\frac{\partial \eta_1}{\partial t}} + \underset{(5)}{\tilde{p}_2\frac{\partial \eta_2}{\partial t}} - \underset{(4)}{\tilde{p}_1\frac{\partial \eta_2}{\partial t}} - \underset{(8)}{\tilde{p}_2\frac{\partial \eta_3}{\partial t}}. \qquad (17.22)$$

Substituting (17.19) - (17.20) into (17.22) yields

$$\frac{\partial}{\partial t}\left(\tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3\right) = \tilde{p}_1\frac{\partial h_1}{\partial t} + \tilde{p}_2\frac{\partial h_2}{\partial t} \qquad (17.23)$$

A comparison of (17.23) and (12.19) shows that the change in total available potential energy for the two active layer, reduced-gravity model does not differ between the layer-layer and the layer-interface formulations.

The change in the total energy per unit area for this case is found by summing (17.9) - (17.14) to obtain

$$\frac{\partial}{\partial t}\left(\tilde{A}_1 + K_1 + \tilde{A}_2 + K_2 + \tilde{A}_3 + K_3\right) = -\vec{\nabla}\cdot\vec{v}_1\left(K_1 + h_1\tilde{p}_1\right)$$

$$-\vec{\nabla}\cdot\vec{v}_2\left(K_2 + h_2\tilde{p}_2\right) \qquad (17.24)$$

or, from (6.28) for N=3,

$$\frac{\partial}{\partial t}\left(\tilde{A}_1 + K_1 + \tilde{A}_2 + K_2 + \tilde{A}_3 + K_3\right) = -\vec{\nabla}\cdot\vec{v}_1\left(K_1 + h_1\hat{p}_1\right)$$

$$-\vec{\nabla}\cdot\vec{v}_2\left(K_2 + h_2\hat{p}_2\right) + \hat{p}_3\frac{\partial \eta_3}{\partial t}. \qquad (17.25)$$

Note that (17.24) and (12.23) are identical as are (17.25) and (12.24) so that the change in total energy per unit area does not depend on the type of formulation: layer-layer or layer-interface.

PART E: Summary and Conclusions

Kinetic energy and potential energy are included in the energy analysis of hydrodynamic models. Internal energy is omitted because these models do not use temperature as a variable. The energy analysis includes the specification of energies present and the change in magnitude of these energies with w.r.t. time. Consider a fluid system that is surrounded by solid walls and bottom. If there are no external sources and sinks of energy, the total of all energies present in the system remains constant with time. However, energy may be redistributed horizontally and vertically within the system and may be transferred from one type to another. The ways in which energy is redistributed and transferred give insight into the behavior of the dynamical model.

Usually less than one percent of the total potential energy can be transferred into other energy types or redistributed in space, so this available potential energy, rather than the total potential energy, is considered in the energy analysis of a fluid system. This available potential energy is defined as the difference between the total potential energy actually present and some minimum potential energy state. Choosing the minimum potential energy state can be difficult for some dynamical models. For the finite depth and reduced-gravity models studied here, the minimum energy state is defined as the state with no motion and no interface deviations.

Both kinetic and available potential energy can be specified per unit volume, but for layered models it seems reasonable to integrate over a layer to find both kinetic and available potential energy per unit area in a layer. This layer specification of energies is used first in section B.1, and is called the layer-layer formulation. However, as shown in section B.3, it is also possible to associate available potential energy with a given interface. The specification of the kinetic energy per unit area in a layer and the interface available potential energy is called the layer-interface formulation. The vertical sorting of available potential energy is different for the two formulations, but the total available potential energy per unit area, found by summing over the vertical extent of the system, is identical for both formulations as it should be.

Once the energies are specified, the change with w.r.t. time of these energies can be found readily from the basic equations as shown in section B.1 for the finite depth model. Then the terms in the energy relations must be grouped carefully to give the energy transfers possible. As shown in section B.4, it is possible to obtain a set of energy transfers that are mathematically correct, but physically meaningless.

For the finite depth model, the layer-layer and the layer-interface formulations both lead to physically reasonable,, but different, energy transfers as shown in Figures 2 and 3. For the layer-layer formulation, transfer between kinetic and available potential energies occurs within a layer, but only available potential energy is exchanged between adjacent layers. For the layer-interface formulation, kinetic energy in a layer is exchanged with interface available potential energy for the interfaces

directly above and below the layer. The interface available potential energy is exchanged with the kinetic energy in the layers directly above and below the interface. The magnitudes of both the kinetic energy per unit area in a layer and its change w.r.t. time are identical for the two formulations. Both formulations offer useful insights into the energetics of the finite depth model, and there appears to be no reason to prefer one over the other.

The basic equations for the finite depth model, together with the assumptions and approximations are found in the Appendix. The reduced-gravity model, discussed in detail in section B.5, has the additional assumption that the bottom layer is infinitely deep. The kinetic energy per unit area for an infinitely deep layer is finite only if the velocity in the layer is zero. The condition of zero velocity, together with conservation of mass, means that the pressure in the bottom layer does not vary with time or horizontal position. Therefore, the interface deviations cannot all be specified independently or, in other words, the interface deviation at the top of the bottom layer is determined by the density structure and the rest of the interface deviations.

In the layer-layer formulation, the same energy transfers are present for the finite depth and reduced-gravity models (Figs. 2 and 5), but the magnitudes of these transfers, particularly for the exchange of available potential energy between adjacent layers, are different. In the layer-interface formulation, the energy transfers for the two models (Figs. 3 and 6) differ in type as well as magnitude. For the reduced-gravity model only, interface available potential energy is exchanged between the interfaces above and below a layer. At first it may appear physically unsatisfactory for interface available potential energy to be exchanged between two interfaces that are not in physical contact. However, this exchange merely reflects the fact that in the reduced-gravity model the interface deviations and, therefore, the interface available potential energies, are coupled and are not all independent. Thus, both formulations are reasonable physically, but the greater complexity of the layer-interface formulation might make the layer-layer formulation more desirable for the reduced-gravity model.

In conclusion, two different,, but physically consistent, formulations of the energetics for the finite depth and the reduced-gravity models have been found.

References

Dutton, J. A. (1976). The Ceaseless Wind. McGraw Hill, New York, 579 pp.

Gill, A. E. (1982). Atmospheric-Ocean Dynamics. Academic Press, New York, 662 pp. (International Geophysics Series, v. 30).

Harrison, D. E. (1979). Eddies and the General Circulation of Numerical Gyres: An Energetic Perspective. Reviews of Geophysics and Space Physics, v. 17, pp. 969-979.

Holland, W. R. and L. B. Lin (1975). On the Generation of Mesoscale Eddies and their Contribution to the Oceanic General Circulation. I. A Preliminary Numerical Experiment. Journal of Physical Oceanography, v. 5, pp. 642-657.

Holton, J. R. (1972). An Introduction to Dynamic Meteorology. Springer-Verlag, New York, 710 pp.

Hurlburt, H. E. and J. D. Thompson (1980). A Numerical Study of Loop Current Intrusions and Eddy Shedding. Journal of Physical Oceanography, v. 10, pp. 1611-1651.

Hurlburt, H. E. and J. D. Thompson (1982). The Dynamics of the Loop Current and Shed Eddies in a Numerical Model of the Gulf of Mexico. In Hydrodynamics of Semi-Enclosed Seas, J. C. J. Nihoul (Editor), pp. 243-298, Elsevier, Amsterdam (The Netherlands), 555 pp.

Lorenz, E. N. (1967). The Nature and Theory of the General Circulation of the Atmosphere. World Meteorological Organization, Geneva (Switzerland), 161 pp.

Pedlosky, J. (1987). Geophysical Fluid Dynamics. Springer-Verlag, New York, 710 pp.

Phillips, O. M. (1977). The Dynamics of the Upper Ocean. University of Cambridge Press, Cambridge (England), 336 pp.

Robinson, A. R., D. W. Harrison and D. B. Haidvogel (1979). Mesoscale Eddies and General Ocean Circulation Models. Dynamics of Oceans and Atmospheres, v. 3, pp. 143-180.

Semtner, A. J. and Y. Mintz (1977). Numerical Simulation of the Gulf Stream and Mid-Ocean Eddies. Journal of Physical Oceanography, v. 7, pp. 208-230.

Appendix: Derivation of the basic equations

Consider the equations for conservation of mass and momentum in a multilayer hydrodynamic ocean model. Assume the fluid is inviscid, hydrostatic, and rotating. In addition, assume that the density in each layer is held constant. The resulting equations for conservation of mass and momentum in a given layer, i, are

$$\left(\frac{\partial u_i}{\partial t}\right) + u_i\left(\frac{\partial u_i}{\partial x}\right) + v_i\left(\frac{\partial u_i}{\partial y}\right) + w_i\left(\frac{\partial u_i}{\partial z}\right) - fv_i = -\frac{1}{\rho_i}\left(\frac{\partial p_i}{\partial x}\right), \tag{A.1}$$

$$\left(\frac{\partial v_i}{\partial t}\right) + u_i\left(\frac{\partial v_i}{\partial x}\right) + v_i\left(\frac{\partial v_i}{\partial y}\right) + w_i\left(\frac{\partial v_i}{\partial z}\right) + fu_i = -\frac{1}{\rho_i}\left(\frac{\partial p_i}{\partial y}\right), \tag{A.2}$$

$$0 = -\frac{1}{\rho_i}\left(\frac{\partial p_i}{\partial z}\right) - g, \tag{A.3}$$

$$\left(\frac{\partial u_i}{\partial x}\right) + \left(\frac{\partial v_i}{\partial y}\right) + \left(\frac{\partial w_i}{\partial z}\right) = 0. \tag{A.4}$$

Subscripts indicate the layer; other symbols are defined in the Glossary. Figure 1 is a vertical cross section of the multilayer model. The pressure at depth z can be obtained by integrating (A.3) vertically from depth z to the top of the ocean:

$$\int_z^{\eta_1}\left(\frac{\partial p_i}{\partial z}\right)dz = -\int_z^{\eta_1}\rho_i g\,dz \tag{A.5}$$

or

$$p_i(x,y,z,t) = p_A(x,y,t) + \rho_i g\left(\eta_1 - z\right) - \sum_{j=1}^{i}\left(\rho_i - \rho_j\right)gh_j \tag{A.6}$$

The applied atmospheric pressure, $p_A(x,y,t)$, will be set to zero and ignored for these calculations. Note that the only z dependence on the (RHS) of (A.6) occurs in the $\rho_i gz$ term. Thus, (A.6) may be written as

$$p_i(x,y,z,t) = \hat{p}_i(x,y,t) - \rho_i gz, \tag{A.7}$$

where

$$\hat{p}_i(x,y,t) = \rho_i g\eta_1 - \sum_{j=1}^{i} \left(\rho_i - \rho_j\right) g h_j \ . \qquad (A.8)$$

Operating on (A.7) with $\vec{\nabla}$, where $\vec{\nabla} \equiv \hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y}$, yields

$$\vec{\nabla}\, p_i(x,y,z,t) = \vec{\nabla}\, \hat{p}_i(x,y,t) \ . \qquad (A.9)$$

Thus, there is no z dependence on the RHS of (A.1) and (A.2). It follows that if the horizontal velocity is independent of z initially, it will remain that way. Then (A.1) and (A.2) become

$$\frac{\partial u_i}{\partial t} + u_i\left(\frac{\partial u_i}{\partial x}\right) + v_i\left(\frac{\partial u_i}{\partial y}\right) - f v_i = -\ \frac{1}{\rho_i}\left(\frac{\partial p_i}{\partial x}\right) \ . \qquad (A.10)$$

$$\frac{\partial v_i}{\partial t} + u_i\left(\frac{\partial v_i}{\partial x}\right) + v_i\left(\frac{\partial v_i}{\partial y}\right) + f u_i = -\ \frac{1}{\rho_i}\left(\frac{\partial p_i}{\partial y}\right) \ . \qquad (A.11)$$

Integrating equation (A.4) over the thickness of a layer yields

$$\int_{\tilde{z}_i}^{\tilde{z}_{i-1}} \left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\right) dz = -\int_{\tilde{z}_i}^{\tilde{z}_{i-1}} \left(\frac{\partial w_i}{\partial z}\right) dz \ . \qquad (A.12)$$

The horizontal velocity does not vary with z within a layer, so (A.12) becomes

$$\left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\right)\left(\tilde{z}_{i-1} - \tilde{z}_i\right) = -\ w_i\left(\tilde{z}_{i-1}\right) + w_i\left(\tilde{z}_i\right) \qquad (A.13)$$

Now

$$w_i\left(\tilde{z}_i\right) \equiv \frac{d\tilde{z}_i}{dt} \qquad (A.14)$$

and

$$\tilde{z}_{i-1} - \tilde{z}_i \equiv h_i \ . \qquad (A.15)$$

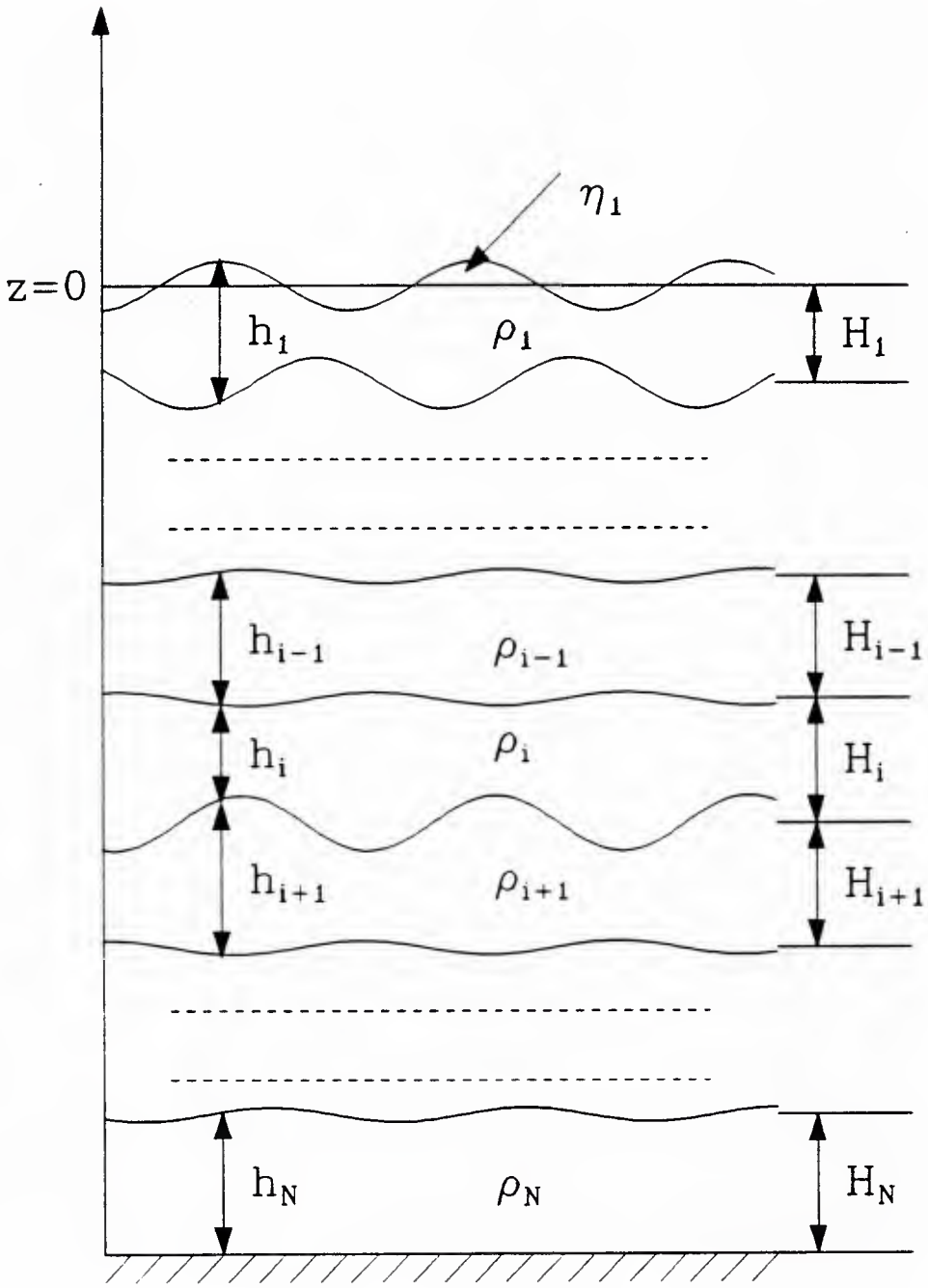
Using (A.14) and (A.15) in (A.13) yields

$$\left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\right) h_i = - \frac{dh_i}{dt} \tag{A.16}$$

or

$$\frac{\partial h_i}{\partial t} + u_i \left(\frac{\partial h_i}{\partial x}\right) + v_i \left(\frac{\partial h_i}{\partial y}\right) + h_i \left(\frac{\partial u_i}{\partial x}\right) + h_i \left(\frac{\partial v_i}{\partial y}\right) = 0 \; . \tag{A.17}$$

(A.6), (A.10), (A.11) and (A.17) form the set of four equations and four unknowns for each layer of the various layered ocean models examined in this study. This set will be used to derive the energy transformations for each model and is listed for easy reference in Table 1 on page 4.




$$\tilde{z}_o = \eta_1$$

$$h_1 = \eta_1 + H_1 - \eta_2$$

$$\tilde{z}_1 = \eta_2 - H_1$$

$$\tilde{z}_{i-1} = \eta_i - \sum_{j=1}^{i-1} H_j$$

$$h_i = \eta_i + H_i - \eta_{i+1}$$

$$\tilde{z}_i = \eta_{i+1} - \sum_{j=1}^{i} H_j$$

$$\tilde{z}_{N-1} = \eta_N - \sum_{j=1}^{N-1} H_j$$

$$h_N = \eta_N + H_N$$

$$\tilde{z}_N = - \sum_{j=1}^{N} H_j$$

Figure 1. A vertical cross section of an N-layer, finite depth model.

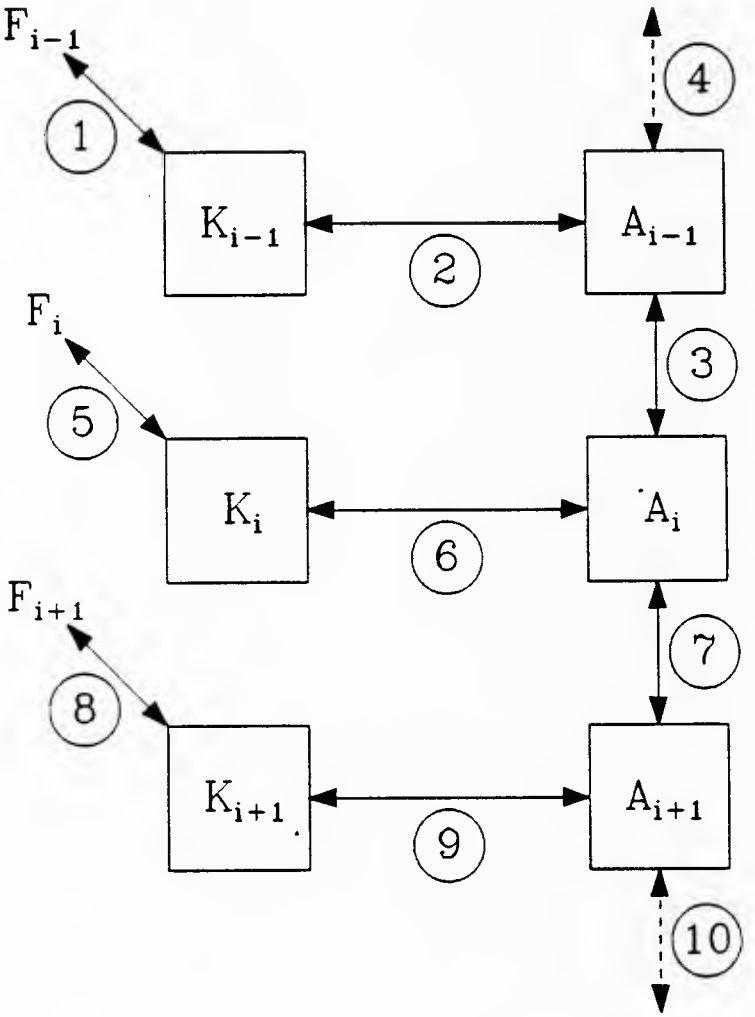



Figure 2. Energy transfers for three adjacent layers in an N-layer, finite depth model for the layer-layer formulation.

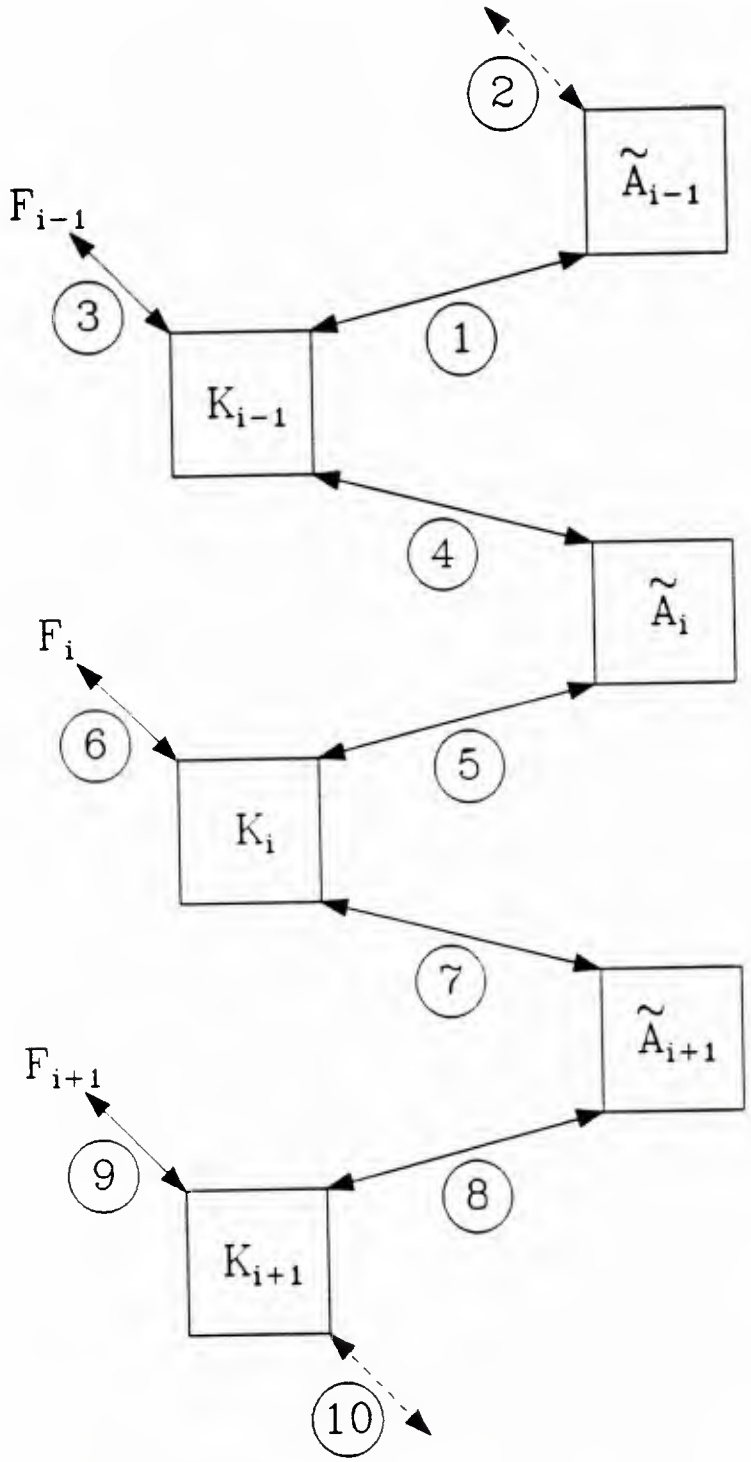



Figure 3. Energy transfers for three adjacent layers and interfaces in an N-layer, finite depth model for the layer-interface formulation.

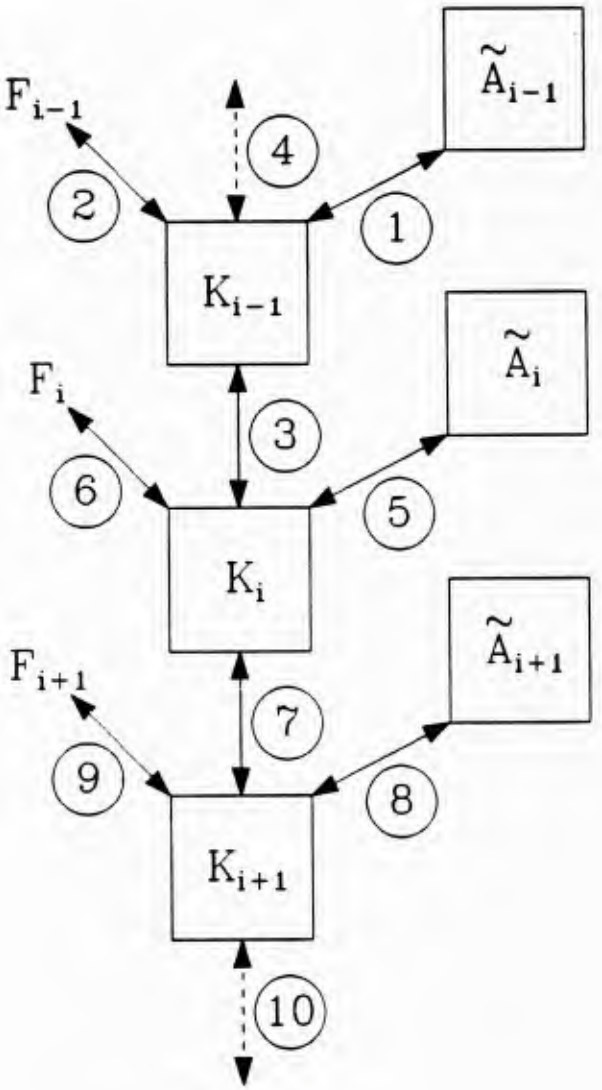



Figure 4. Energy transfers for three adjacent layers and interfaces in an N-layer, finite depth model for a physically unrealistic formulation.

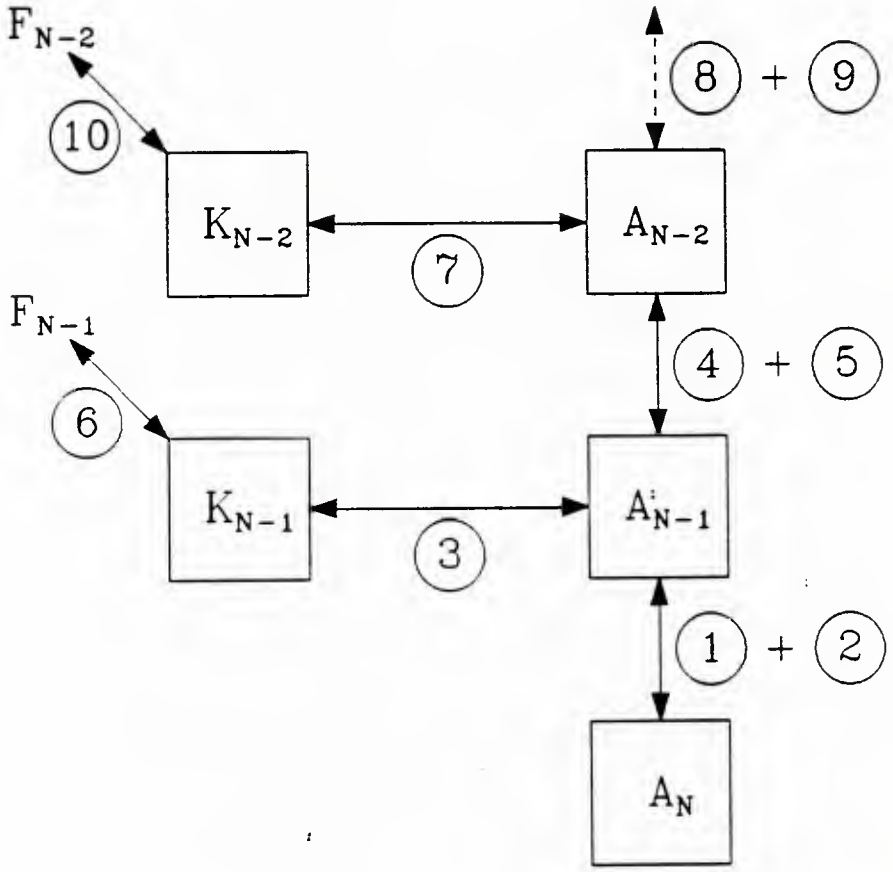



Figure 5. Energy transfers for the bottom three layers in an N-layer, reduced-gravity model for the layer-layer formulation.

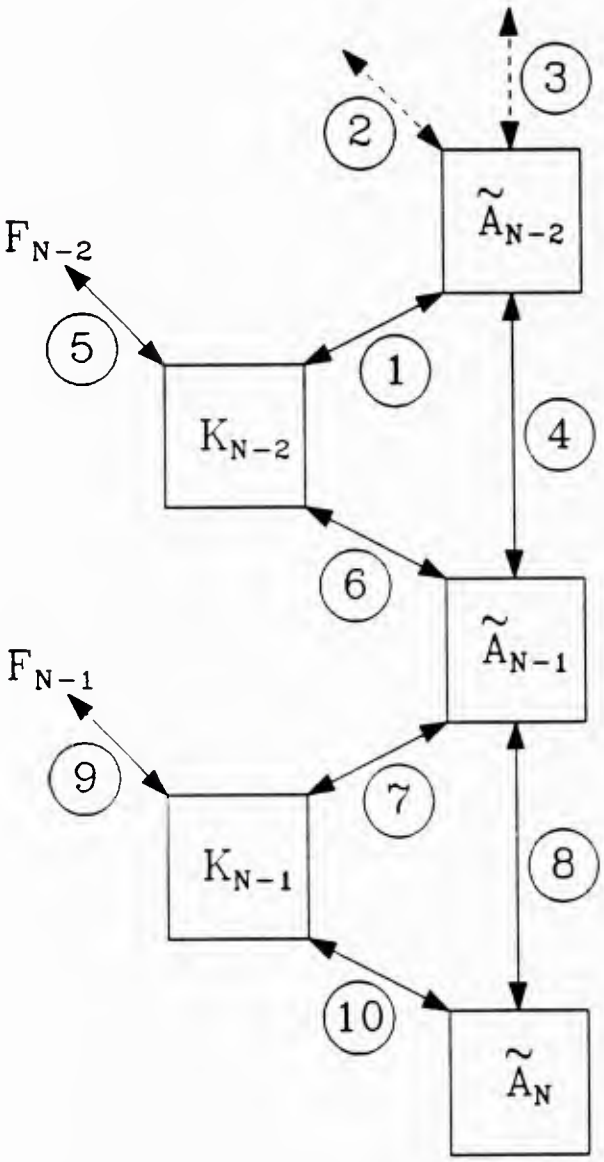



Figure 6. Energy transfers for the bottom three layers and interfaces in an N-layer, reduced-gravity model for the layer-interface formulation.

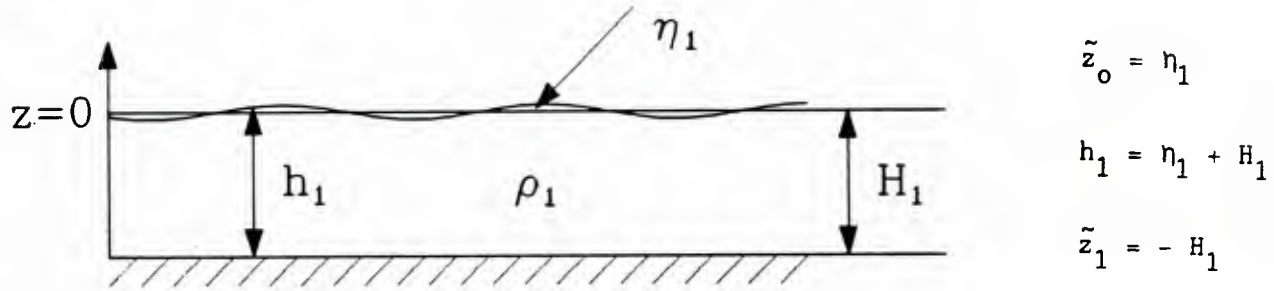



Figure 7. A vertical cross-section of a one-layer, finite depth model.

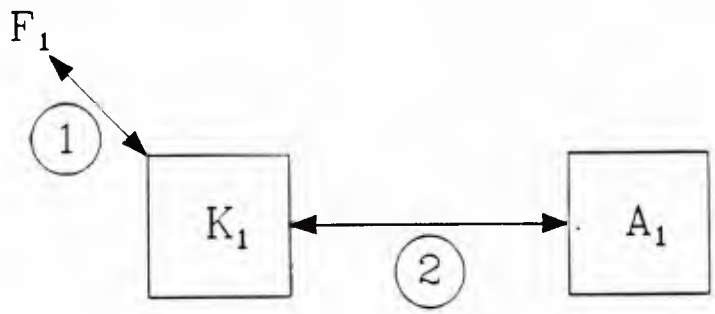



Figure 8. Energy transfers in a one-layer, finite depth model for the layer-layer formulation.

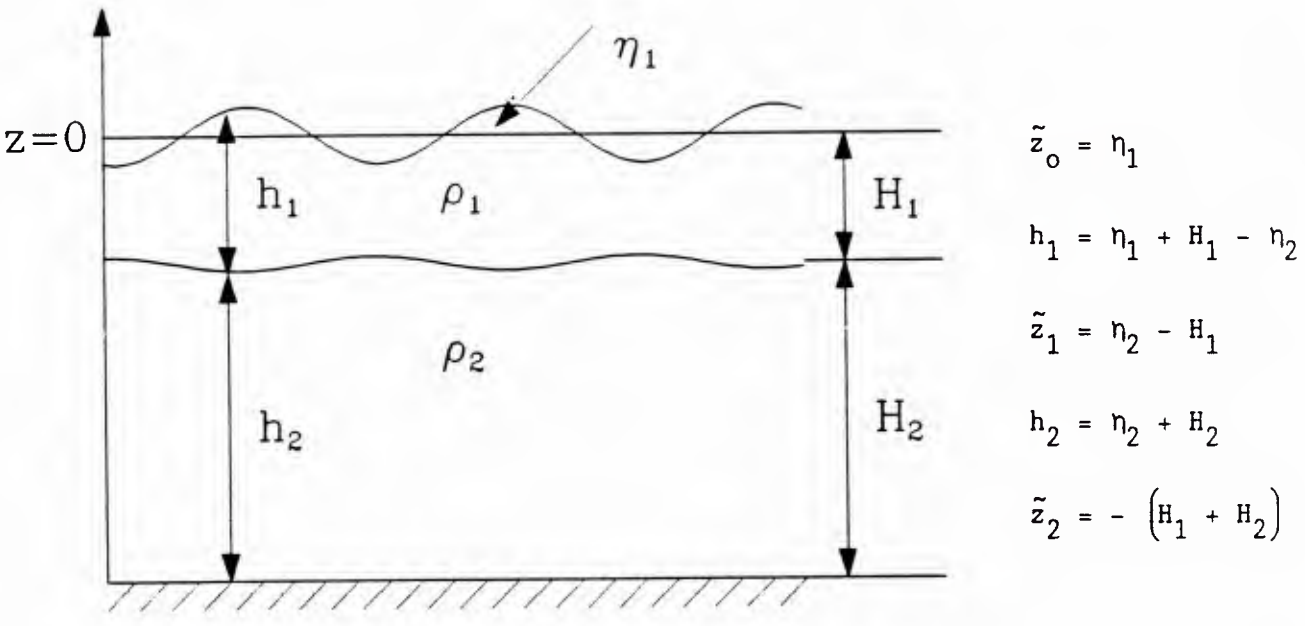



Figure 9. A vertical cross-section of a two-layer, finite depth model.

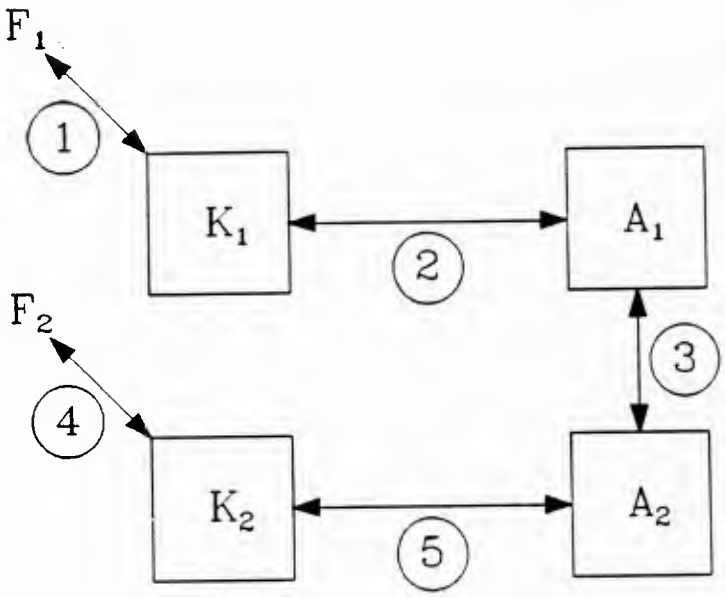



Figure 10. Energy transfers in a two-layer, finite depth model for the layer-layer formulation.

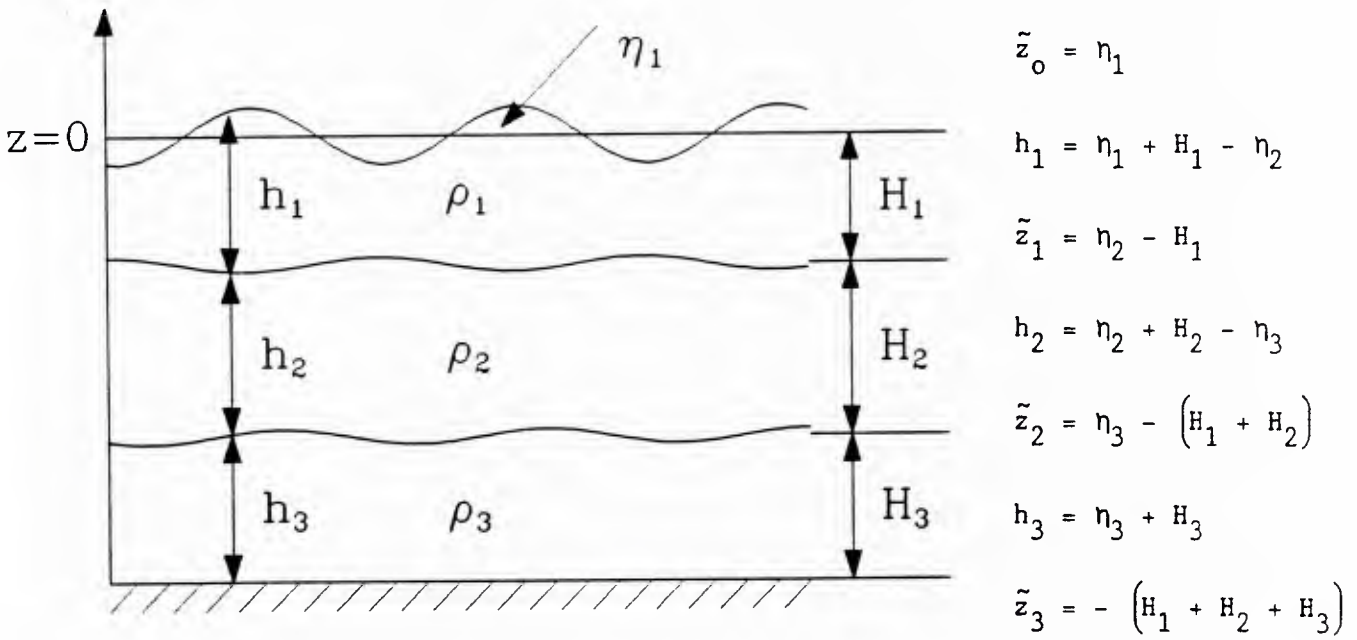



Figure 11. A vertical cross section of a three-layer, finite depth model.

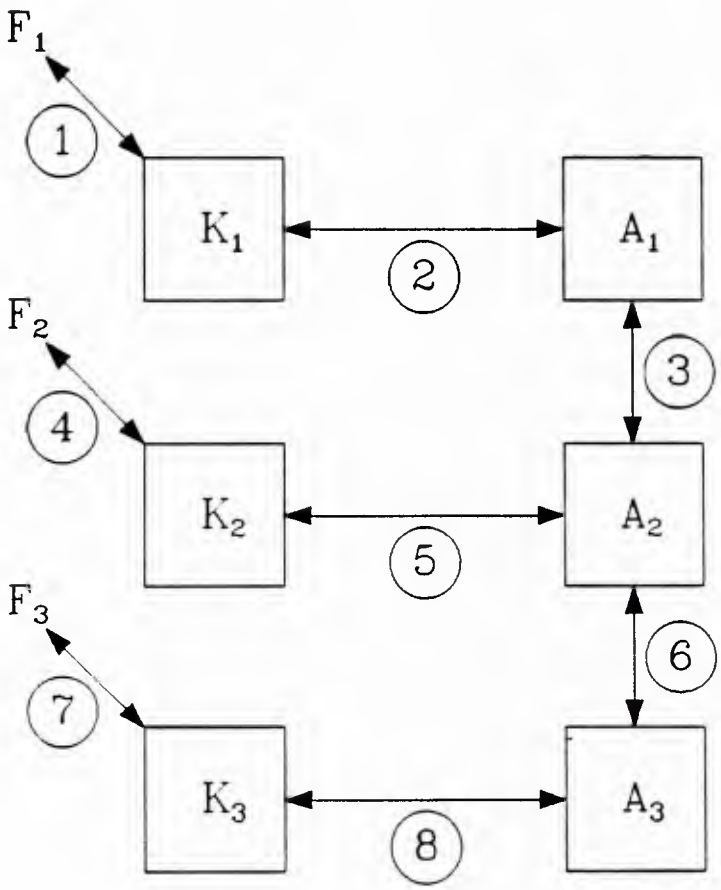
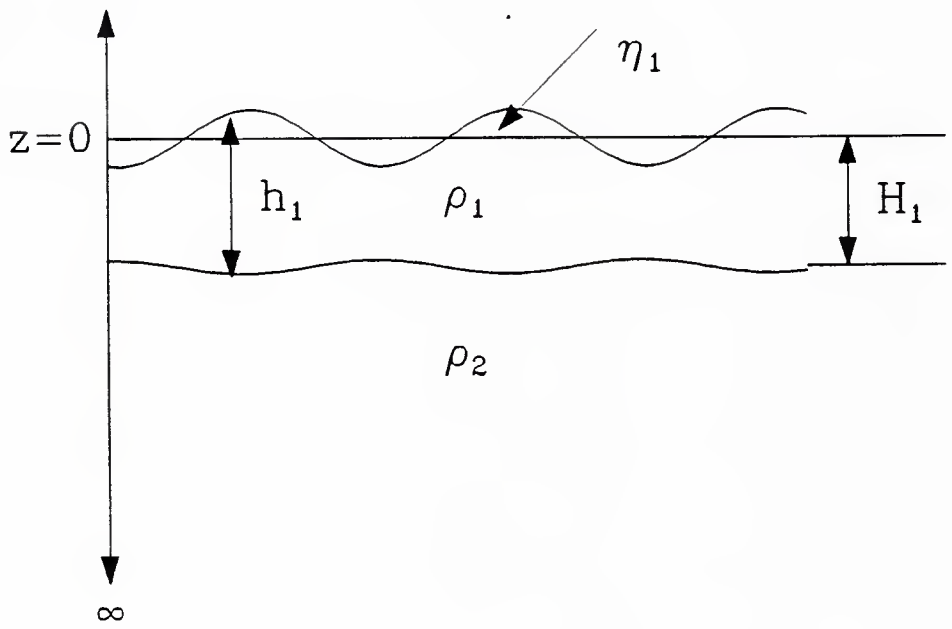



Figure 12. Energy transfers in a three-layer, finite depth model for the layer-layer formulation.




$\tilde{z}_o = \eta_1$

$h_1 = \eta_1 + H_1 - \eta_2$

$\tilde{z}_1 = \eta_2 - H_1$

Figure 13. A vertical cross section of a one active layer, reduced-gravity model.

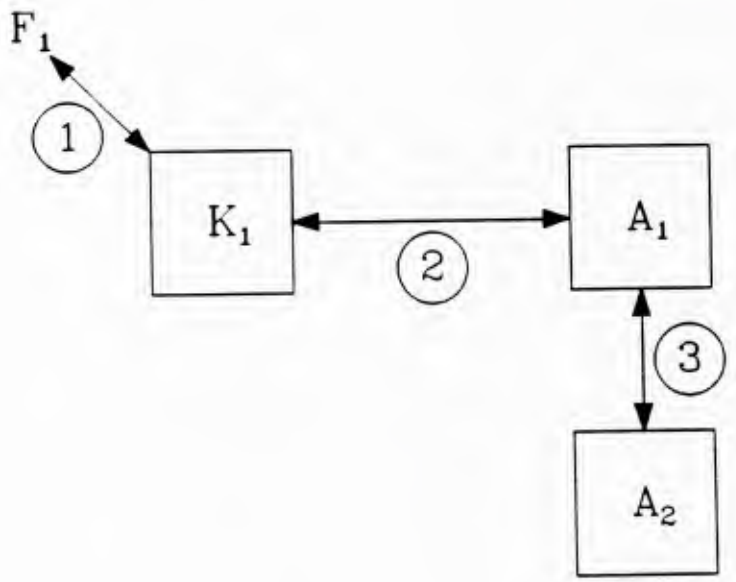



Figure 14. Energy transfers in a one active layer, reduced-gravity model for the layer-layer formulation.

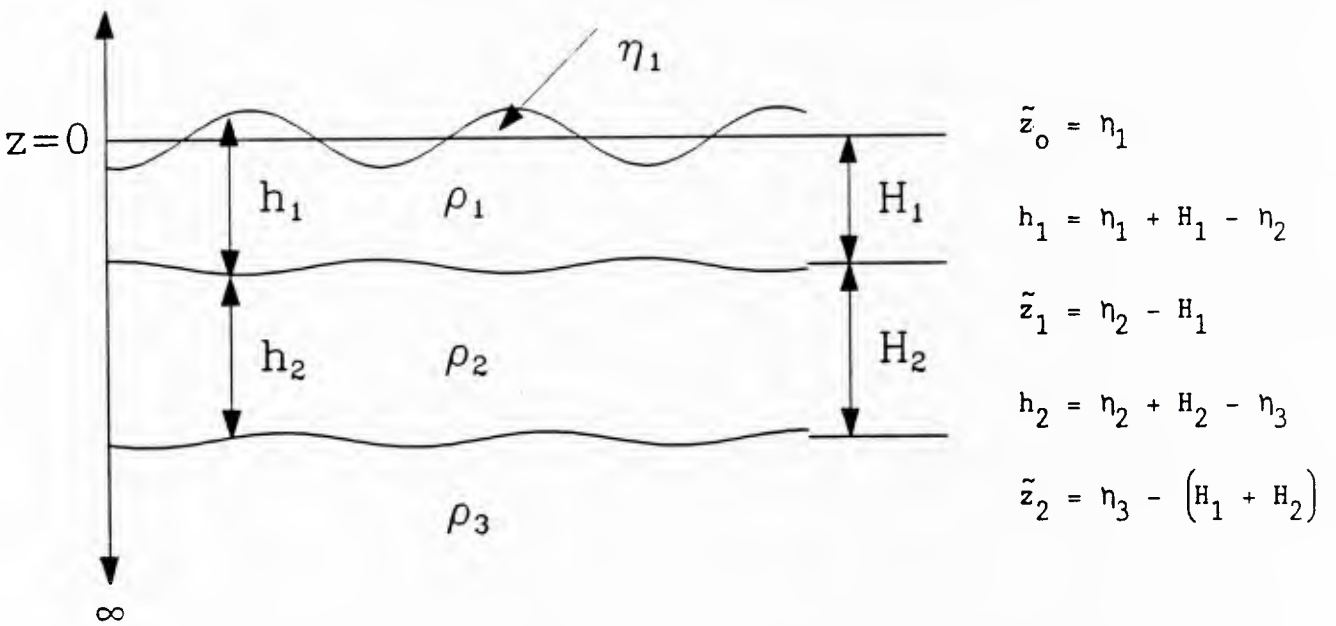



Figure 15. A vertical cross section of a two active layer, reduced-gravity model.

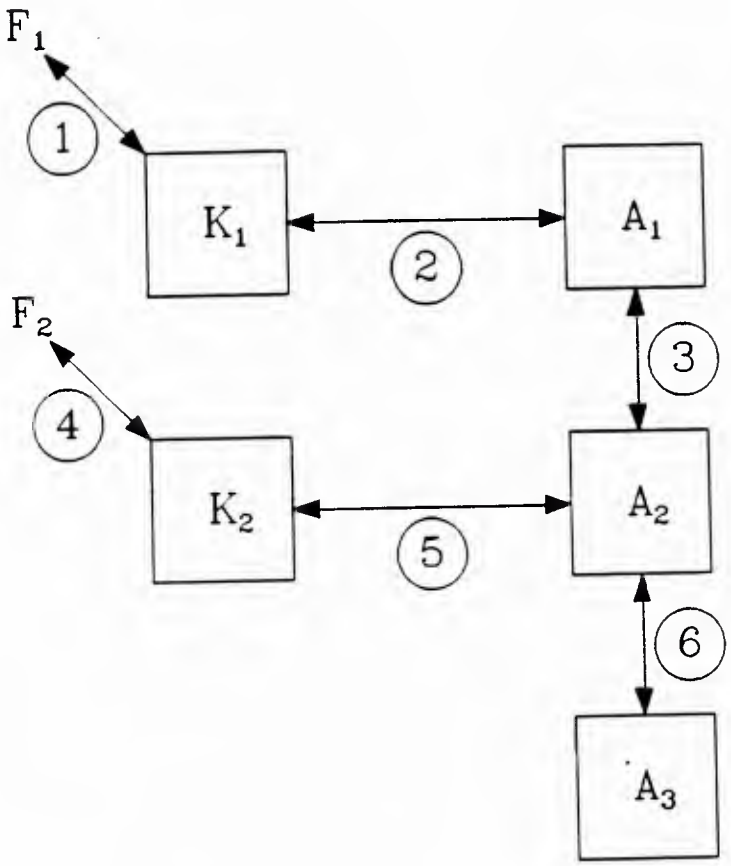



Figure 16. Energy transfers in a two active layer, reduced-gravity model for the layer-layer formulation.

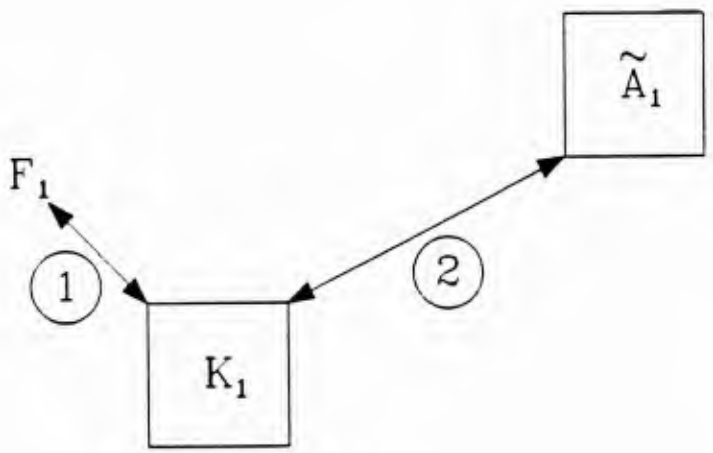



Figure 17. Energy transfers in a one-layer, finite depth model for the layer-interface formulation.

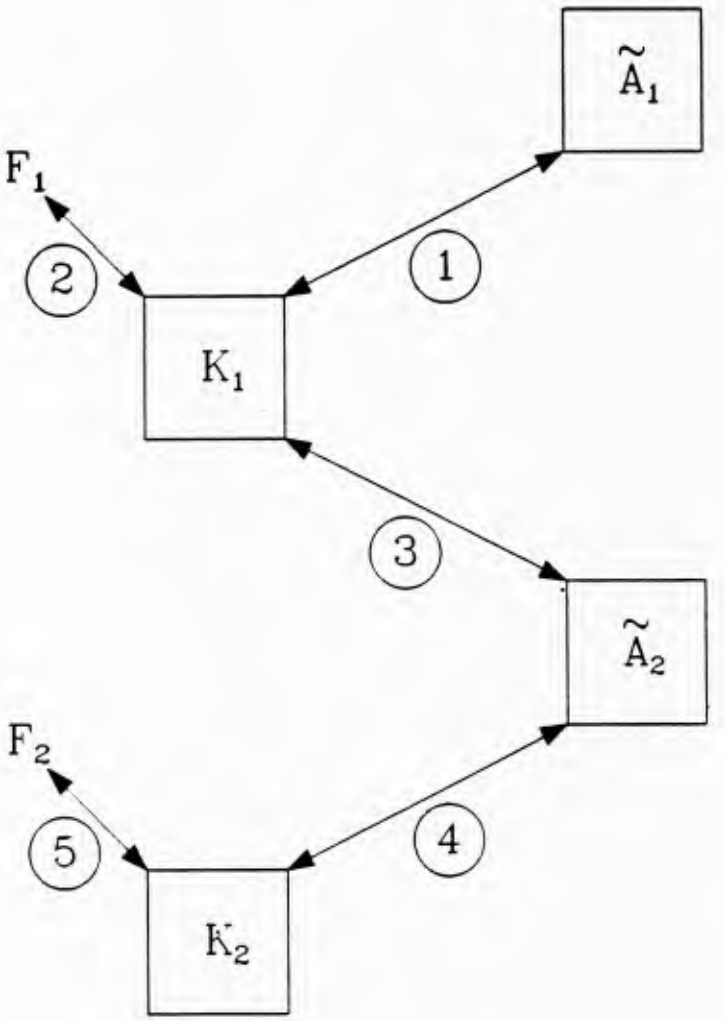



Figure 18. Energy transfers in a two-layer, finite depth model for the layer-interface formulation.

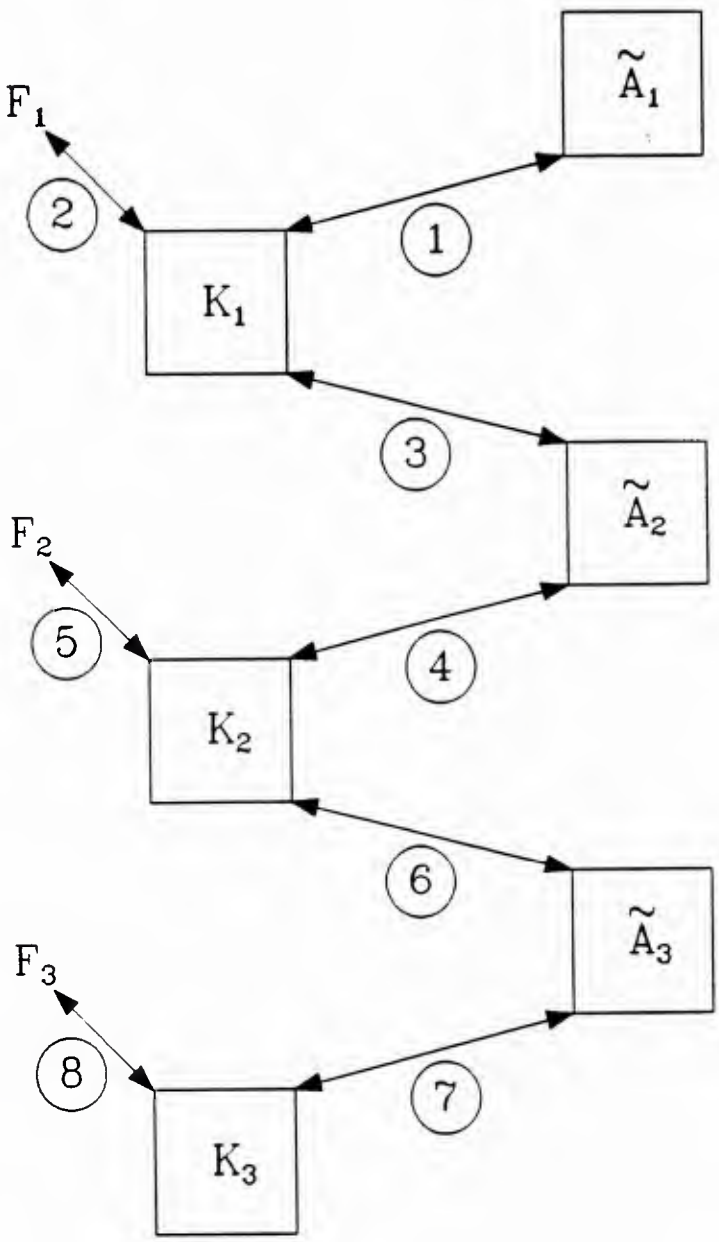



Figure 19. Energy transfers in a three-layer, finite depth model for the layer-interface formulation.

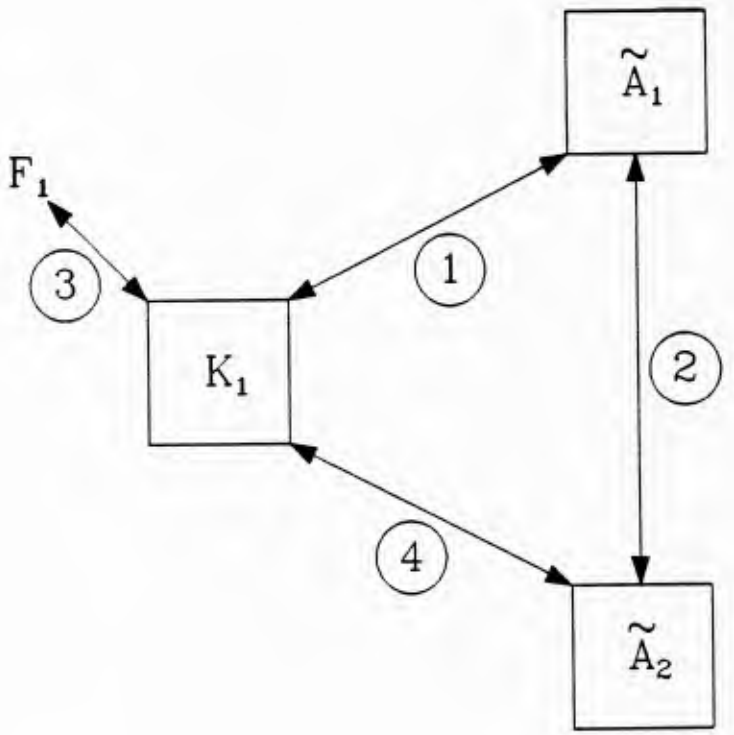



Figure 20. Energy transfers in a one active layer, reduced-gravity model for the layer-interface formulation.

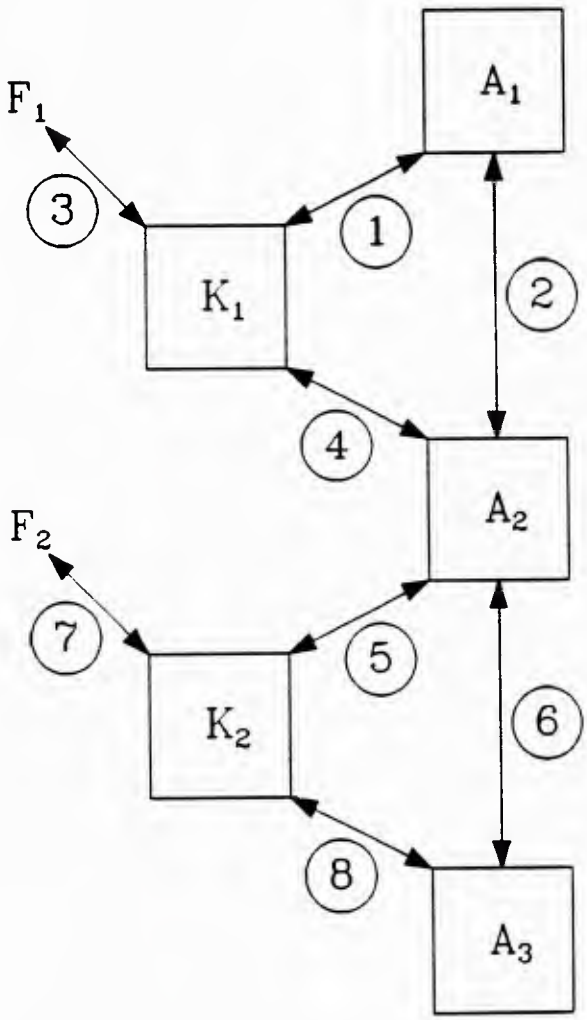



Figure 21. Energy transfers in a two active layer, reduced-gravity model for the layer-interface formulation.